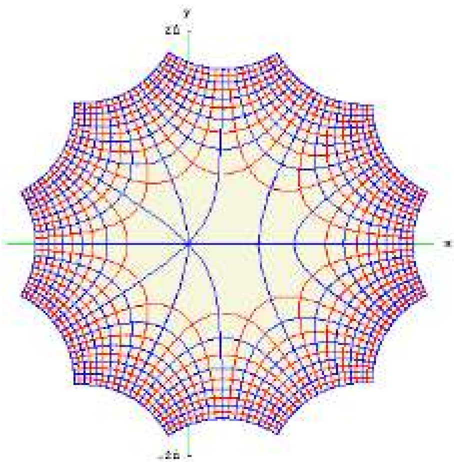
Complex Analysis

Lecture notes

Dr. rer. Nat. Muhammad Mustahsan
University of Stuttgart, Germany.



Lecture 1. Algebra of complex Numbers

The symbol i was originally used as a disguise for the embarrassing symbol $\sqrt{-1}$. W e now say that i is the **imaginary unit** and define it by the property $i^2 = -1$. Using the imaginary unit, we build a general complex number out of two real numbers.

A **complex number** is any number of the form $z = a + ib$ where a and b are real numbers and i is the imaginary unit. The notations $a + ib$ and $a + bi$ are used interchangeably. The real number a in $z = a + ib$ is called the **real part** of z; the real number b is called the **imaginary part** of z. The real and imaginary parts of a complex number z are abbreviated $Re(z)$ and $Im(z)$, respectively.

Complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are **equal**, $z_1 = z_2$, if $a_1 = a_2$ and $b_1 = b_2$.

Complex numbers can be added, subtracted, multiplied, and divided. If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, these operations are defined as follows.

Addition: $z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$

Subtraction: $z_1 - z_2 = (a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i(b_1 - b_2)$

Multiplication:

$$z_1 \ldots {}_1 a_2 - b_1 b_2) + i\left(a_2 b_1 + a_1 b_2\right)$$

Division:

$$z_1 / z_2 = \left(a_1 + i b_1\right) / \left(a_2 + i b_2\right) = \left[\left(a_1 a_2 + b_1 b_2\right) + i\left(a_2 b_1 - a_1 b_2\right)\right] / \left({a^2}_2 + {b^2}_2\right)$$

The familiar commutative, associative, and distributive laws hold for complex numbers:

Commutative laws: $\begin{cases} z_1 + z_2 = z_2 + z_1 \\ z_1 z_2 = z_2 z_1 \end{cases}$

Associative laws: $\begin{cases} z_1 + \left(z_2 + z_3\right) = \left(z_1 + z_2\right) + z_3 \\ z_1\left(z_2 z_3\right) = \left(z_1 z_2\right) z_3 \end{cases}$

Distributive law: $z_1\left(z_2 + z_3\right) = z_1 z_2 + z_1 z_3$

Addition, Subtraction, and Multiplication

(i) To add (subtract) two complex numbers, simply add (subtract) the corresponding real and imaginary parts.

(ii) To multiply two complex numbers, use the distributive law and the fact that $i^2 = -1$.

The **zero** in the complex number system is the number 0 + 0i and the **unity** is 1 + 0i. The zero and unity are denoted by 0 and 1, respectively. The zero is the **additive identity** in the complex number system since, for any complex number $z = a + ib$, we have $z + 0 = z$. To see this, we use the definition of addition:

$z + 0 = (a + ib) + (0 + 0i) = a + 0 + i(b + 0) = a + ib = z$.

Similarly, the unity is the **multiplicative identity** of the system since, for any complex number z, we have $z \cdot 1 = z \cdot (1 + 0i) = z$.

If z is a complex number, the number obtained by changing the sign of its imaginary part is called the **complex conjugate**, or simply **conjugate**, of z and is denoted by the symbol $\bar{z}$.In other words, if $z = a + ib$,then its conjugate is $\bar{z} = a - ib$.

From the definitions of addition and subtraction of complex numbers, it is readily shown that the conjugate of a sum and difference of two complex numbers is the sum and difference of the conjugates:

$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2,\ \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2.$

Moreover, we have the following three additional properties:

$\overline{z_1 z_2} = \bar{z}_1\ \bar{z}_2,\ \ \overline{z_1 / z_2} = \bar{z}_1\ / \bar{z}_2,$

Of course, the conjugate of any finite sum (product) of complex numbers is the sum (product) of the conjugates.

The definitions of addition and multiplication show that the sum and product of a complex number z with its conjugate $\bar{z}$ is a real number:

$$z + \bar{z} = (a + ib) + (a - ib) = 2a$$

$$z\bar{z} = (a + ib)(a - ib) = a^2 - i^2b^2 = a^2 + b^2.$$

The difference of a complex number z with its conjugate $\bar{z}$ is a pure imaginary number:

$$z - \bar{z} = (a + ib) - (a - ib) = 2ib$$

Since a = Re(z) and b = Im(z), these yield two useful formulas:

$Re(z) = \dfrac{z + \bar{z}}{2}$ and $Im(z) = \dfrac{z - \bar{z}}{2i}$

However, (4) is the important relationship in this discussion because it enables us to approach division in a practical manner.

Division

To divide z_1 by z_2, multiply the numerator and denominator of z_1/z_2 by the conjugate of z_2. That is,

$$\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}} = \frac{z_1\overline{z_2}}{|z_2|^2}$$

and then use the fact that $z_1\overline{z_2}$ is the sum of the squares of the real and imaginary parts of z_2.

In the complex number system, every number z has a unique **additive inverse**. As in the real number system, the additive inverse of z = a + ib is its negative, −z, where −z = −a − ib. For any complex number z, we have z + (−z) = 0. Similarly , every nonzero complex number z has a **multiplicative inverse**. In symbols, for $z \neq 0$ there exists one and only one nonzero complex number z^{-1} such that $zz^{-1} = 1$.The multiplicative inverse $z^{-1}$is the same as the **reciprocal** 1/z.

Comparison with Real Analysis

(i) Many of the properties of the real number system **R** hold in the complex number system **C**, but there are some truly remarkable differences as well. For example, the concept of order in the real number system does not carry over to the complex number system. In other words, we cannot compare two complex numbers $z_1 = a_1 + ib_1, b_1 \neq 0$, and $z_2 = a_2 + ib_2, b_2 \neq 0$, by means of inequalities. Statements such as $z_1 < z_2$ or $z_2 \geq z_1$ have no meaning in **C** except in the special case when the two numbers z_1 and z_2 are real. Therefore, if you see a statement such as $z_1 = \alpha z_2$, $\alpha > 0$, it is implicit from the use of the inequality $\alpha > 0$ that the symbol α represents a real number.

(ii) Some things that we take for granted as impossible in real analysis, such as $e^x = -2$ and $\sin x = 5$ when x is a real variable, are perfectly correct and ordinary in complex analysis when the symbol x is interpreted as a complex variable.

By using the definitions of addition, multiplication and distributive laws, one can easily prove that the set of complex number is a **field**.

Assignment:

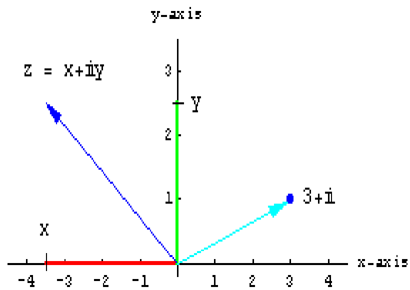
1. What can be said about the complex number z if $z = \bar{z}$? If $z^2 = (\bar{z})^2$?
2. Assume that $\sqrt{1+i}$ make sense. How would you demonstrate the validity of the equality $\sqrt{1+i} = \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{2}} + i\sqrt{-\frac{1}{2}+\frac{1}{2}\sqrt{2}}$?
3. Suppose z_1 and z_2 are complex numbers. What can be said about z_1 or z_2 if $z_1z_2 = 0$?
4. Suppose the product z_1z_2 of two complex numbers is a nonzero real constant. Show that $z_2 = k\overline{z_1}$, where k is a real number.
5. Without doing any significant work, explain why it follows immediately that $z_1\bar{z}_2 + \bar{z}_1z_2 = 2\mathrm{Re}(z_1\bar{z}_2)$.

Lecture 2. Geometry of complex numbers (Rectangular form)

A complex number $z = x + iy$ is uniquely determined by an ordered pair of real numbers (x, y).The first and second entries of the ordered pairs correspond, in turn, with the real and imaginary parts of the complex number.

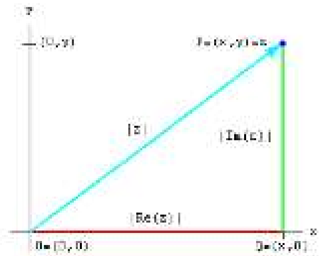
The coordinate plane is called the **complex plane** or simply the **z - plane**. The horizontal or x-axis is called the **real axis** because each point on that axis represents a real number. The vertical or y-axis is called the **imaginary axis** because a point on that axis represents a pure imaginary number.

Thus, a complex number $z = x + iy$ can also be viewed as a two dimensional position vector, that is, a vector whose initial point is the origin and whose terminal point is the point (x, y).



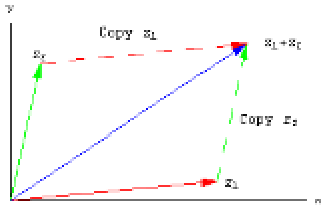
The modulus of a complex number $z = x + iy$, is the real number

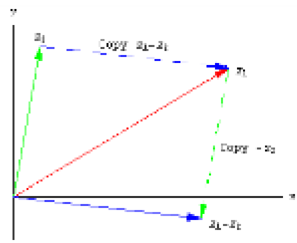
$$|z| = x^2 + y^2$$



The modulus |z| of a complex number z is also called the **absolute value** of z. We shall use both words modulus and absolute value throughout this text.

The vector interpretation of the sum $z_1 + z_2$ is the vector shown in Figure as the main diagonal of a parallelogram whose initial point is the origin and terminal point is $(x_1 + x_2, y_1 + y_2)$.The difference $z_2 - z_1$ can be drawn either starting from the terminal point of z_1 and ending at the terminal point of z_2, or as a position vector whose initial point is the origin and terminal point is $(x_2 - x_1, y_2 - y_1)$.





In the case $z = z_2 - z_1$, it follows from (1) that the **distance between two points** $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ in the complex plane is the same as the distance between the origin and the point $(x_2 - x_1, y_2 - y_1)$; that is, $|z| = |z_2 - z_1| = |(x_2 - x_1) + i(y_2 - y_1)|$ or

$$|z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example: Describe the set of points z in the complex plane that satisfy $|z| = |z - i|$.

We know from geometry that the length of the side of the triangle corresponding to the vector $z_1 + z_2$ cannot be longer than the sum of the lengths of the remaining two sides. In symbols we can express this observation by the inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

The result in (6) is known as the **triangle inequality**. Now from the identity

$$z_1 = z_1 + z_2 + (-z_2),$$

gives

$$|z_1| = |z_1 + z_2 + (-z_2)| \leq |z_1 + z_2| + |-z_2|.$$

Since $|z_2| = |-z_2|$, solving the last result for $|z_1 + z_2|$ yields another important inequality:

$$|z_1 + z_2| \geq |z_1| - |z_2|.$$

But because $z_1 + z_2 = z_2 + z_1$, can be written in the alternative form

$|z_1 + z_2| = |z_2 + z_1| \geq |z_2| - |z_1| = -(|z_1| - |z_2|)$ and so combined with the last result implies

$$|z_1 + z_2| \geq |z_1| - |z_2|$$

It also follows from (6) by replacing z_2 by $-z_2$ that

$$|z_1 + (-z_2)| \leq |z_1| + |(-z_2)| = |z_1| + |z_2|.$$

This result is the same as

$$|z_1 - z_2| \leq |z_1| + |z_2|\,.$$

From (8) with z_2 replaced by $-z_2$, we also find

$$|z_1 - z_2| \geq ||z_1| - |z_2||.$$

Assignments

1. How would you describe geometrically the relationship between a nonzero complex number $z = a + ib$ and its

(a) negative, $-z$?

(b) inverse, z^{-1}?

2. Under what circumstances does $|z_1 + z_2| = |z_1| + |z_2|$?

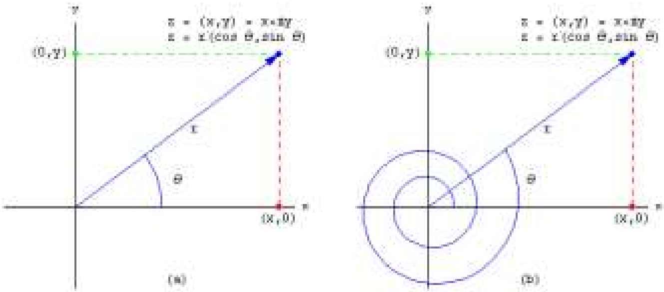
3. Consider the complex numbers $z_1 = 4 + i, z_2 = -2 + i, z_3 = -2 - 2i, z_4 = 3 - 5i$.

(**a**) Use four different sketches to plot the four pairs of points z_1, iz_1; z_2, iz_2; z_3, iz_3; and z_4, iz_4

(**b**) In general, how would you describe geometrically the effect of multiplying a complex number $z = x + iy$ by i? By $-i$?

Lecture 3. Geometry of complex numbers

A point P in the plane whose rectangular coordinates are (x, y) can also be described in terms of polar coordinates. The polar coordinate system, invented by Isaac Newton, consists of point O called the pole and the horizontal half-line emanating from the pole called the polar axis. If r is a directed distance from the pole to P and θ is an angle of inclination (in radians) measured from the polar axis to the line OP, then the point can be described by the ordered pair (r, θ), called the polar coordinates of P.



Then x, y, r and θ are related by $x = r\cos\theta, y = r\sin\theta$.These equations enable us to express a nonzero complex number $z = x + iy$ as $z = (r\cos\theta) + i(r\sin\theta)$ or $z = r(\cos\theta + i\sin\theta)$. We say that (1) is the **polar form** or **polar representation** of the complex number z.

In other words, we shall adopt the convention that r is never negative so that we can take r to be the modulus of z, that is, $r =$

$|z|$. The angle θ of inclination of the vector z, which will always be measured in radians from the positive real axis, is positive when measured counterclockwise and negative when measured clockwise.

The angle θ is called an **argument** of z and is denoted by $\theta = \arg(z)$. An argument θ of a complex number must satisfy the equations $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$. An argument of a complex number z is not unique since $\cos\theta$ and $\sin\theta$ are 2π $-$periodic; in other words, if θ_0 is an argument of z, then necessarily the angles $\theta_0 + 2\pi, \theta_0 + 4\pi, \ldots$ are also arguments of z. In practice we use $\tan\theta = \frac{y}{x}$ to find θ. The symbol arg(z) actually represents a set of values, but the argument θ of a complex number that lies in the interval $-\pi < \theta \le \pi$ is called the **principal value** of arg(z) or the **principal argument** of z. The principal argument of z is unique and is represented by the symbol Arg(z), that is,

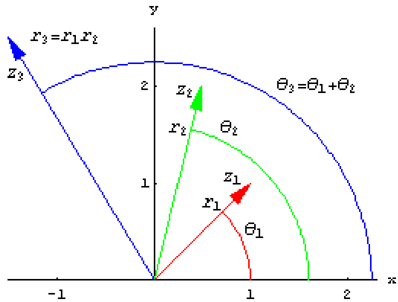
$-\pi < Arg(z) \le \pi.$

The polar form of a complex number is especially convenient when multiplying or dividing two complex numbers.

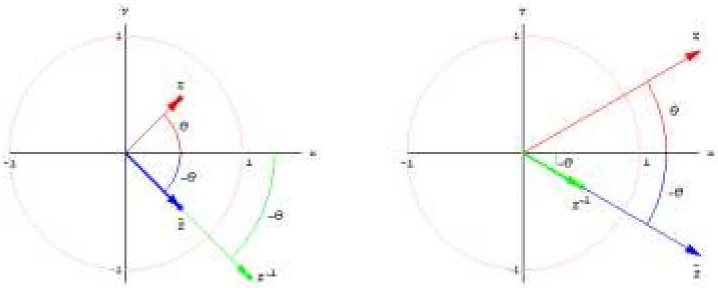
Suppose $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, where θ_1 and θ_2 are any arguments of z_1 and z_2, respectively. Then $z_1 z_2 = r_1 r_2\,[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\,(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$ (4)

and, for $z_2 \neq 0$,

$z_1/z_2 \ = \ r_{1/}r_2[\cos\theta_1 \cos\theta_2 \ + \ \sin\theta_1 \sin\theta_2 \ +$
$i\,(\sin\theta_1 \cos\theta_2 \ - \ \cos\theta_1 \sin\theta_2)]$. (5)



From the addition formulas for the cosine and sine, it can be rewritten as $z_1z_2 \ = \ r_1r_2\,[\cos(\theta_1 \ + \ \theta_2) \ + \ i\sin(\theta_1 \ + \ \theta_2)]$ (6) and $z_1/z_2 = \ r_1/r_2[\cos(\theta_1 \ - \ \theta_2) \ + \ i\sin(\theta_1 \ - \ \theta_2)]$.



Inspection of the expressions and Figure shows that the lengths of the two vectors z_1z_2 and z_1/z_2 are the product of the lengths of z_1 and z_2 and the quotient of the lengths of z_1 and z_2 , respectively. Moreover, the arguments of z_1z_2 and z_1/z_2 are given by $arg(z_1\,z_2) = \ arg(z_1) + \ arg(z_2) and$ $\arg(\,z_1\,z_2) = \ arg(z_1) \ -$ $arg(z_2)$.

We can find integer powers of a complex number z from the results. For example, if $z = r(\cos\theta + i\sin\theta)$, then with $z_1 = z_2 = z$, (6) gives

$z^2 = r^2 [\cos(\theta + \theta) + i\sin(\theta + \theta)] = r^2 (\cos 2\theta + i\sin 2\theta)$. Since $z^3 = z^2 z$, it then follows that $z^3 = r^3 (\cos 3\theta + i\sin 3\theta)$, and so on. In addition, if we take $\arg(1) = 0$, then (7) gives

$\frac{1}{z^2} = z^{-2} = r^{-2}[\cos(-2\theta) + i\sin(-2\theta)]$.

Continuing in this manner, we obtain a formula for the nth power of z for any integer n:

$z^n = r^n (\cos n\theta + i\sin n\theta)$. (9)

When n = 0, we get the familiar result $z^0 = 1$. When z = cos θ +i sin θ, we have |z| = r = 1, and so (9) yields

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. (10)

This last result is known as **de Moivre's formula** and is useful in deriving certain trigonometric identities involving $\cos n\theta$ and $\sin n\theta$.

When $z = \cos\theta + i\sin\theta$, we have $|z| = r = 1$, and so (9) yields

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. (10)

This last result is known as **de Moivre's formula** and is useful in deriving certain trigonometric identities involving $\cos n\theta$ and $\sin n\theta$.

Suppose $z = r(\cos\theta + i\sin\theta)$ and $w = \rho(\cos\varphi + i\sin\varphi)$ are polar forms of the complex numbers z and w. Then, the equation $w^n = z$ becomes

$\rho^n(\cos n\varphi + i\sin n\varphi) = r(\cos\theta + i\sin\theta).$ (1)

From (1), we can conclude that $\rho^n = r$ (2) and $\cos n\varphi + i\sin n\varphi = \cos\theta + i\sin\theta.$ (3)

From (2), we define $\rho = \sqrt[n]{r}$ to be the unique positive nth root of the positive real number r. From (3), the definition of equality of two complex numbers implies that

$\cos n\varphi = \cos\theta$ and $\sin n\varphi = \sin\theta$.

These equalities, in turn, indicate that the arguments θ and φ are related by $n\varphi = \theta + 2k\pi$, where k is an integer. Thus,

$$\varphi = \frac{\theta + 2k\pi}{n}$$

As k takes on the successive integer values $k = 0, 1, 2, \dots, n-1$ we obtain n distinct nth roots of z; these roots have the same modulus $\sqrt[n]{r}$ but different arguments. Notice that for $k \geq n$ we obtain the same roots because the sine and cosine are 2π-periodic.

To see why this is so, suppose $k = n + m$, where $m = 0, 1, 2, \dots$.Then

$$\varphi = \frac{\theta + 2(n + m)\pi}{n} = \frac{\theta + 2m\pi}{n} + 2\pi$$

and $\sin\varphi = \sin(\frac{\theta + 2m\pi}{n})$, $\cos\varphi = \cos(\frac{\theta + 2m\pi}{n})$.

We summarize this result. The n nth roots of a nonzero complex number

$z = r(\cos\theta + i\sin\theta)$ are given by

$w_k = n\sqrt{r}(\cos(\frac{\theta + 2m\pi}{n}) + i\sin(\frac{\theta + 2m\pi}{n}))$, (4)

where $k = 0, 1, 2, \ldots, n - 1$.

Assignment

1. Find the three cube roots of $z = i$.
2. Find the four fourth roots of $z = 1 + i$.
3. A real number can have a complex nth root. Can a non real complex number have a real nth root?
4. Suppose w is located in the first quadrant and is a cube root of a complex number z. Can there exist a second cube root of z located in the first quadrant? Defend your answer with sound mathematics.
5. Use the fact that $8i = (2 + 2i)^2$ to find all solutions of the equation $z^2 - 8z + 16 = 8i$.
6. The n distinct **nth roots of unity** are the solutions of the equation $w^n = 1$.

Lecture 4. Functions of complex variables

A complex-valued function f(z) of the complex variable z is a rule that assigns to each complex number z in a set D one and only one complex number w. We write w = f(z) and call w the image of z under w = f(z). The set D is called the domain of f(z), and the set of all images {w = f(z): z ∈ D} is called the range of f(z). When the context is obvious, we omit the phrase "complex-valued," and simply refer to a complex function f(z) . We can define the domain to be any set that makes sense for a given rule, so for $w = f(z) = z^2$, we could have the entire complex plane for the domain D, or we might artificially restrict the domain to some set such as $D = D_1(0) = \{z : |z| < 1\}$. Determining the range for a function defined by a formula is not always easy, but we will see plenty of examples later on.

Cartesian Coordinate Form

Just as z can be expressed by its real and imaginary parts = x + iy , we write $w = f(z) = u + iv$, where u and v are the real and imaginary parts of w, respectively. Doing so gives us the representation $w = f(z) = f(x, y) = u + iv$.

Because u and v depend on x and y, they can be considered to be real-valued functions of the real variables x and y; that is, $u = u(x, y), v = v(x, y)$.

Combining these ideas, we often write a complex function f(z), in the form $w = f(z) = f(x, y) = f(x + iy) = u(x, y) + iv(x, y)$.

Polar Coordinate Form

Using the $z = re^{i\theta}$, that was developed, in the expression of a complex function f(z) may be convenient. It gives us the polar coordinate representation

$$f(z) = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta),$$

where $u(r, \theta)$ and $v(r, \theta)$ are real functions of the real variables r and θ.

Remark. For a given function f(z), the functions $u(r, \theta)$ and $v(r, \theta)$ defined above are different from those defined previously in Equation $f(z) = f(x, y) = f(x + iy) = u(x, y) + iv(x, y)$, because Equation involves Cartesian coordinates and equation involves polar coordinates.

We now look at the geometric interpretation of a complex function. If D is the domain of real-valued functions and u(x,y) and v(x,y), then the system of equations u(x,y) and v(x,y), describes a transformation (or mapping) from D in the xy-plane into the uv-plane, also called the w-plane. Therefore, we consider the function

$$w = f(z) = f(x, y) = f(x + iy) = u(x, y) + iv(x, y).,$$

to be a transformation (or mapping) from the set D in the z-plane onto the range R in the w-plane. In the following paragraphs we present some additional key ideas (in blue). If A is a subset of the domain D of $w = f(z)$, then the set $B = \{w = f(z): z \in A\}$ is called the image of the set A, and $w = f(z)$ is said to map A onto B. The image of a single point is a single point, and the image of the entire domain, D, is the range, R. The mapping $w = f(z)$ is said to be from A into S if the image of A is contained in S. Mathematicians use the notation $f: A \to S$ to indicate that a function maps A into S.

Definition. (One-To-One Function) A function $w = f(z)$ is said to be one-to-one if it maps distinct points $z_1 \neq z_2$ onto distinct points $f(z_1) \neq f(z_2)$.

Definition. (Inverse Function) Given the function $w = f(z)$, an inverse function $z = g(w)$ will satisfy the following equations, $g(f(z)) = z$ for all $z \in A$, and $f(g(w)) = w$ for all $z \in B$.

Furthermore, if $w = f(z)$ and $z = g(w)$ are functions that map A into B and B into A, respectively. We usually indicate the inverse of $w = f(z)$ by the notation $z = f^{-1}(w)$. If the domains of $f(z)$ and $f^{-1}(w)$ are A and B respectively.

Visualizing the Image of a Set

We now show how to find the image B of a specified set A under a given mapping

$u+iv=w=f(x+iy)=f(z)$,

from the z-plane into the w-plane. The set A is usually described with an equation or inequality involving the variables x and y.

Start with $w = f(z)$ and solve for z_1 and get

$$z=f^{-1}(w)$$
$$x+iy=f^{-1}(u+iv)$$
$$x+iy=\phi(u,v)+i\varphi(u,v)$$

Then identify the real and imaginary functions $x=\phi(u,v), y=\varphi(u,v)$. Then use these formulas can be used as substitutions in given formulas that describe set A, in the z-plane. Don't worry about using the notation $x=\phi(u,v), y=\varphi(u,v)$, just jump right in and solve the equations at hand.

Example. Show that the function $f(z) = i z$ maps the line $y = x + 1$ in the xy-plane onto the line $v = -u - 1$ in the uv-plane.

Therefore, the image of $A = \{(x, y) : y = x + 1\}$ under $w = f(z) = i z$, is the set $B = \{(u, v) : v = -u - 1\}$.

Method 2. We write $u + i v = w = f(z) = i(x + i y) = -y + i x$ and note that the transformation can be given by the system of equations $u = -y$ and $v = x$. Because A is described by $A = \{x + i y : y = x + 1\}$, we can substitute $u = -y$ and $v = x$ into the equation $y = x + 1$ to obtain $-u = v + 1$, which we can rewrite as $v = -u - 1$. If you use this method, be sure to pay careful attention to domains and ranges.

Example. Show that the image of the open disk $D_1(-1 - i) = \{z : |z + 1 + i| < 1\}$ under the linear transformation $w = f(z) = (3 - 4i) z + 6 + 2i$, is the open disk

$$D_5(1 - 3i) = \{w : |w + 1 - 3i| < 5\}.$$

Solution. The inverse transformation is $= \frac{w-6-2i}{3-4i}$, so, if the range of $w = f(z)$ is B, then

$$w = f(z) \in B$$

$$\Longleftrightarrow \quad f^{-1}(w) = z \in D_1(-1 - i)$$

$$\Longleftrightarrow \quad \frac{w - 6 - 2i}{3 - 4i} \in D_1(-1-i)$$

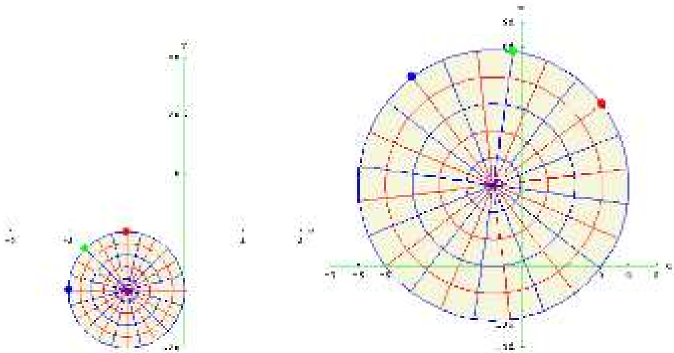
$$\Longleftrightarrow \quad \left| \frac{w - 6 - 2i}{3 - 4i} + 1 + i \right| < 1$$

$$\Longleftrightarrow \quad \left|\frac{w-6-2i}{3-4i}+1+i\right| \, |3-4i| < 1\,|3-4i|$$

$$\Longleftrightarrow \quad |w-6-2i+(1+i)(3-4i)| < 5$$

$$\Longleftrightarrow \quad |w+1-3i| < 5$$

Hence the disk with center $-1-i$ and radius 1 is mapped one-to-one and onto the disk with center $-1+3i$ and radius 5, as shown in Figure.

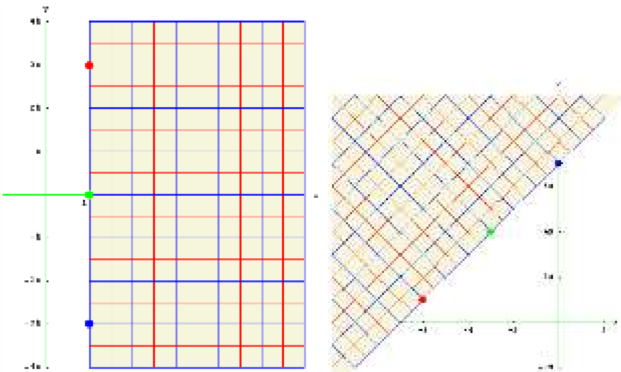


Example. Show that the image of the right half plane $\mathrm{Re}(z) = x \geq 1$ under the linear transformation $w = f(z) = (-1+i)z - 2+3i$, is the half plane $v \geq u+7$.

Solution. The inverse transformation is given by

$$z = \frac{w + 2 - 3i}{-1 + i} = \frac{u + 2 + i(v - 3)}{-1 + i},$$

Substituting $x = \frac{-u + v - 5}{2}$ into $\operatorname{Re}(z) = x \geq 1$ gives $\frac{-u + v - 5}{2} \geq 1$, which simplifies as $v \geq u + 7$.



Assignment:

Lecture 5. Limit of complex functions, continuity and differentiability

Suppose $z_0 = x_0 + iy_0$. Since

$$|z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

is the distance between the points $z = x + iy$ and $z_0 = x_0 + iy_0$, the points $z = x + iy$ that satisfy the equation

$|z - z_0| = \rho, \rho > 0,$

The points z that satisfy the inequality $|z - z_0| \leq \rho$ can be either on the circle $|z - z_0| = \rho$ or within the circle. We say that the set of points defined by $|z - z_0| \leq \rho$ is a **disk** of radius ρ centered at z_0.But the points z that satisfy the strict inequality $|z - z_0| < \rho$ lie within, and not on, a circle of radius ρ centered at the point z_0.This set is called a **neighborhood** of z_0**.** Occasionally, we will need to use a neighborhood of z_0 that also excludes z_0**.** Such a neighborhood is defined by the simultaneous inequality $0 < |z - z_0| < \rho$ and is called a **deleted neighborhood** of z_0.

A point z_0 is said to be an **interior point** of a set S of the complex plane if there exists some neighborhood of z_0 that lies entirely within S. If every point z of a set S is an interior point, then S is said to be an **open set**. The set S1 of points satisfying the inequality $\rho_1 < |z - z_0|$ lie exterior to the circle of radius ρ_1 centered at z_0, whereas the set S2 of points satisfying $|z - z_0| < \rho_2$ lie interior to the circle of radius ρ_2 centered at z_0. Thus, if $0 < \rho_1 < \rho_2$, the set of points satisfying the simultaneous inequality

$$\rho_1 < |z - z_0| < \rho_2, \qquad (2)$$

is the intersection of the sets S1 and S2.This intersection is an open circular ring centered at z_0.Figure illustrates such a ring centered at the origin. The set defined by (2) is called an open **circular annulus**.

If any pair of points z1 and z2 in a set S can be connected by a polygonal line that consists of a finite number of line segments joined end to end that lies entirely in the set, then the set S is said to be connected. An open connected set is called a domain.

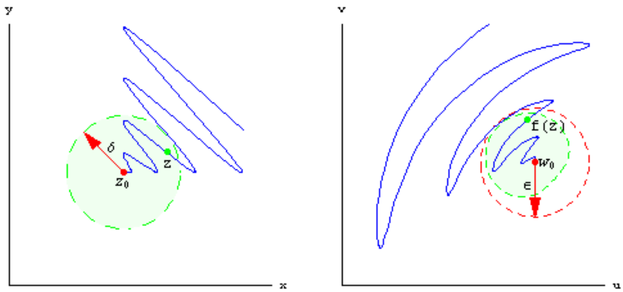
A **region** is a set of points in the complex plane with all, some, or none of its boundary points. Since an open set does not contain any boundary points, it is automatically a region. A region that contains all its boundary points is said to be **closed**. The disk defined by $|z - z_0| \leq \rho$ is an example of a closed region and is referred to as a **closed disk**. A neighborhood of a point z_0 defined by $|z - z_0| < \rho$ is an open set or an open region and is said to be an **open disk**.If the center z_0 is deleted from either a closed disk or an open disk, the regions defined by $0 < |z - z_0| \leq \rho$ or $0 < |z - z_0| < \rho$ are called **punctured disks**. A punctured open disk is the same as a deleted neighborhood of z_0.A region can be neither open nor closed.

Definition: Limit of a Complex Function

Suppose that a complex function f is defined in a deleted neighborhood of z_0 and suppose that L is a complex number. The

limit of f as z tends to z_0 exists and is equal to L, written as $\lim_{z \to z_0} f(z) = L$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever

$$0 < |z - z_0| < \delta.$$



Criterion for the Nonexistence of a Limit

If f approaches two complex numbers $L1 \neq L2$ for two different curves or paths through z_0, then $\lim_{z \to z_0} f(z)$ does not exist.

Suppose that $f(z) = u(x, y) + iv(x, y), z_0 = x_0 + iy_0$, and $L = u_0 + iv_0$. Then $\lim_{z \to z_0} f(z) = L$ if and only if

$$\lim(x, y) \to (x_0, y_0)u(x, y) = u_0 \text{ and } \lim(x, y) \to (x_0, y_0)v(x, y) = v_0.$$

Continuity of complex function

The definition of continuity for a complex function is, in essence, the same as that for a real function. That is, a complex function f is continuous at a point z_0 if the limit of f as z approaches z_0 exists and is the same as the value of f at z_0.

A complex function f is **continuous at a point** z_0 if

$$\lim z \to z_0\, f(z) = f(z_0).$$

Criteria for Continuity at a Point

A complex function f is continuous at a point z_0 if each of the following three conditions hold:

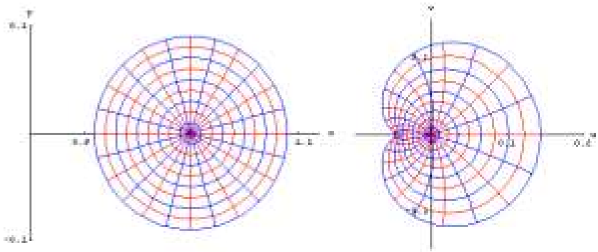
(i) $\lim z \to z_0 f(z)$ exists,

(ii) f is defined at z_0, and

(iii) $\lim z \to z_0 f(z) = f(z_0)$.

Suppose that $f(z) = u(x,y) + iv(x,y)$ and $z_0 = x_0 + iy_0$. Then the complex function f is continuous at the point z_0 if and only if both real functions u and v are continuous at the point (x_0, y_0).

Example: Show that the polynomial $P(z) = 1 - z - z^2 + z^3 - z^4 + z^5$ is continuous at the point $z_0 = 1$ in the complex plane.



Assignment

1. (**a**)Is it true that $\lim z \to z_0 \overline{f(z)} = \lim z \to z_0 f(\overline{z})$ for any complex function f? If so, then give a brief justification; if not, then find a counter example.

   (**b**) If f(z) is a continuous function at z0, then is it true that f(z) is continuous at z_0?

2. If f is a function for which $lim x \to 0 f(x + i0) = 0$ and $lim y \to 0 f(0 + iy) = 0$, then can you conclude that $lim z \to 0 f(z) = 0$? Explain.

Lecture 6. Differentiability and analyticity of complex function

Using our imagination, we take our lead from elementary calculus and define the derivative of f(z) at z_0, written $f'(z_0)$ by

$$f'(z_0) = \lim_{z \to z_0} \left\{ \frac{f(z) - f(z_0)}{z - z_0} \right\},$$

provided that the limit exists. If it does, we say that the function $f(z)$ is differentiable at z0. If we write $\Delta z = z - z0$, then we can express as

$$f'(z_0) = \lim_{\Delta z \to 0} \left\{ \frac{f(z + \Delta z) - f(z_0)}{\Delta z} \right\} .$$

Example. Use the limit definition to find the derivative of $f(z) = z^3$.

Example. Show that the function $f(z) = \bar{z}$ is nowhere differentiable.

Definition 3.1 (Analytic Function). The complex function $f(z)$ is analytic at the point z_0 provided there is some $\varepsilon > 0$ such that $f'(z)$ exists for all $z \in D_\varepsilon(z_0)$. In other words, $f(z)$ must be differentiable not only at z_0, but also at all points in some ε-neighborhood of z_0.

If $f(z)$ is analytic at each point in the region R, then we say that $f(z)$ is an analytic function on R. Again, we have a special term if $f(z)$ is analytic on the whole complex plane.

Definition 3.2 (Entire Function). If $f(z)$ is analytic on the whole complex plane then $f(z)$ is said to be an entire function.

Points of non-analyticity for a function are called singular points.

Graphical explorations of difference quotients.

Example 2. Consider the real function

$$f(x) = \frac{x^6}{6} + 2x^3,$$

which is differentiable, and it's derivative is the limit of the real difference quotients $\frac{f(x+\Delta x) - f(x)}{\Delta x}$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{1}{6}(x+\Delta x)^6 + 2(x+\Delta x)^3 - \frac{1}{6}x^6 - 2x^3}{\Delta x}$$

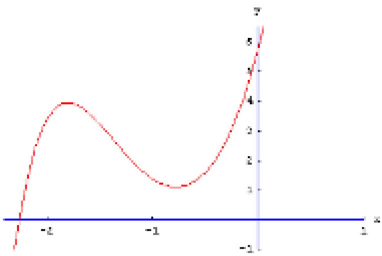
$$= \lim_{\Delta x \to 0} \frac{(x^5+6x^2)\Delta x + \left(\frac{5x^4}{2}+6x\right)\Delta x^2 + \left(\frac{10x^3}{3}+2\right)\Delta x^3 + \frac{5x^2}{2}\Delta x^4 + x\,\Delta x^5 + \frac{1}{6}\Delta x^6}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(x^5 + 6x^2 + \left(\frac{5x^4}{2}+6x\right)\Delta x + \left(\frac{10x^3}{3}+2\right)\Delta x^2 + \frac{5x^2}{2}\Delta x^3 + x\,\Delta x^4 + \frac{1}{6}\Delta x^5\right)$$

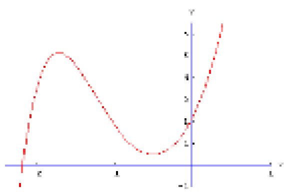
$$= x^5 + 6x^2$$

We can illustrate convergence of the real difference quotients $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$ by comparing graphs for decreasing values of Δx. For illustration purposes we plot the real graphs

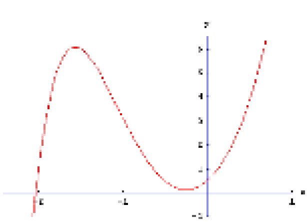
$$y = \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad \text{for} \quad \Delta x = 1.5,\ 1.0,\ 0.5,\ 0.1\,.$$

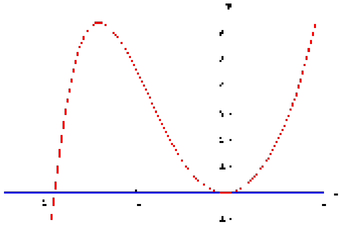


$y = \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$, for $\Delta x = \dfrac{3}{2}$



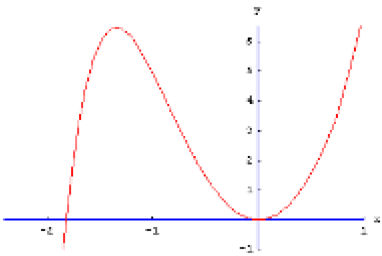
$y = \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$, for $\Delta x = 1$





$$y = \frac{f(x+\Delta x) - f(x)}{\Delta x}, \text{ for } \Delta x = \frac{1}{2}$$

$$y = \frac{f(x+\Delta x) - f(x)}{\Delta x}, \text{ for } \Delta x = \frac{1}{10}$$



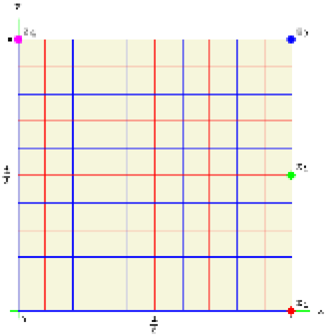
The graph of $y = f'(x) = x^5 + 6x^2$.

where $f(x) = \frac{1}{6}x^6 + 2x^3$ and the graph of $y = f'(x) = x^5 + 6x^2$.

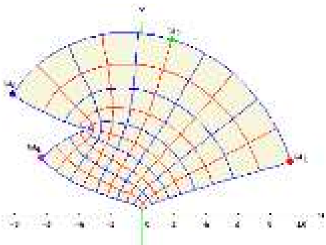
Example 3. Consider the complex function $f(z) = \frac{z^6}{6} + 2z^3$, which is differentiable, and it's derivative is the limit of the complex difference quotients $\frac{f(z+\Delta z) - f(z)}{\Delta z}$.

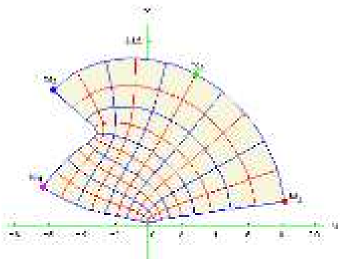
$$
\begin{aligned}
f'(z) &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{\frac{1}{6}(z+\Delta z)^6 + 2(z+\Delta z)^3 - \frac{1}{6}z^6 - 2z^3}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{(z^5 + 6z^2)\Delta z + \left(\frac{5z^4}{2} + 6z\right)\Delta z^2 + \left(\frac{10z^3}{3} + 2\right)\Delta z^3 + \frac{5z^2}{2}\Delta z^4 + z\Delta z^5 + \frac{1}{6}\Delta z^6}{\Delta z} \\
&= \lim_{\Delta z \to 0} \left(z^5 + 6z^2 + \left(\frac{5z^4}{2} + 6z\right)\Delta z + \left(\frac{10z^3}{3} + 2\right)\Delta z^2 + \frac{5z^2}{2}\Delta z^3 + z\Delta z^4 + \frac{1}{6}\Delta z^5\right) \\
&= z^5 + 6z^2
\end{aligned}
$$

We can illustrate convergence of the complex difference quotients $\frac{f(z+\Delta z) - f(z)}{\Delta z}$ by comparing graphs for decreasing values of Δz. For illustration purposes we plot the graphs $w = \frac{f(z+\Delta z) - f(z)}{\Delta z}$ for $\Delta z = 0.4\,\frac{1+i}{\sqrt{2}},\ 0.2\,\frac{1+i}{\sqrt{2}},\ 0.1\,\frac{1+i}{\sqrt{2}},\ 0.05\,\frac{1+i}{\sqrt{2}}$. We cannot draw a graph of 2-dimensional space into 2-dimensional space, it is necessary to choose a domain D in the z-plane for our graphs.



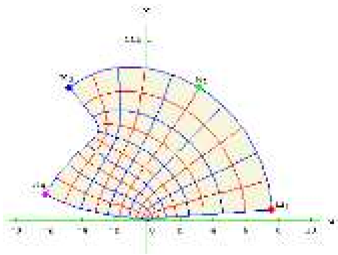
The domain D in the z-plane for the following graphs.

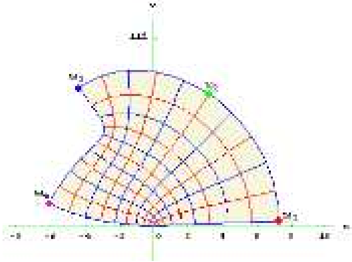




$$w = \frac{f(z+\Delta z) - f(z)}{\Delta z}, \quad \text{for } \Delta z = 0.40\,\frac{1+i}{\sqrt{2}}$$

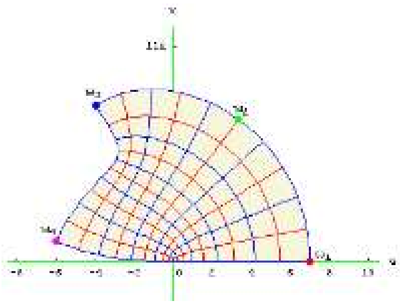
$$w = \frac{f(z+\Delta z) - f(z)}{\Delta z}, \quad \text{for } \Delta z = 0.20\,\frac{1+i}{\sqrt{2}}$$





$$w = \frac{f(z+\Delta z) - f(z)}{\Delta z}, \text{ for } \Delta z = 0.10\,\frac{1+i}{\sqrt{2}}$$

$$w = \frac{f(z+\Delta z) - f(z)}{\Delta z}, \text{ for } \Delta z = 0.05\,\frac{1+i}{\sqrt{2}}$$



The graph of $w = f'(z) = z^5 + 6z^2$.

Figure. The unit square in the z-plane, and it's images under the mappings

$$w = \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad \text{for} \quad \Delta z = 0.4\,\frac{1+i}{\sqrt{2}},\ 0.2\,\frac{1+i}{\sqrt{2}},\ 0.1\,\frac{1+i}{\sqrt{2}},\ 0.05\,\frac{1+i}{\sqrt{2}},$$

where $f(z) = \frac{1}{6} z^6 + 2 z^3$ and the graph of $w = f'(z) = z^5 + 6 z^2$.

Assignment

Suppose $f'(z)$ exists at a point z. Is $f'(z)$ continuous at z?

Lecture 7. Cauchy Riemann equations

Necessary condition

Suppose $f(z) = u(x,y) + iv(x,y)$ is differentiable at a point $z = x + iy$. Then at z the first-order partial derivatives of u and v exist and satisfy the Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $and \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

.

Proof The derivative of f at z is given by

$$f'(z) = lim\Delta z \to 0 \frac{\left(f(z + \Delta z) - f(z)\right)}{\Delta z}$$

By writing $f(z) = u(x,y) + iv(x,y)$ and $\Delta z = \Delta x + i\Delta y$, (2) becomes

$$f'(z) = lim\Delta z \to 0 \frac{\left(u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x,y) - iv(x,y)\right)}{\Delta x + i\Delta y}$$

.

Since the limit is assumed to exist, Δz can approach zero from any convenient direction. In particular, if we choose to let Δz → 0 along a horizontal line, then Δy = 0 and Δz = Δx. We can then write as

$$f'(z) = lim\Delta x \to 0 \frac{(u(x + \Delta x, y) - u(x,y) + i\,[v(x + \Delta x, y) - v(x,y)])}{\Delta x}$$

$$= lim\Delta x \to 0 \frac{\left(u(x + \Delta x, y) - u(x,y)\right)}{\Delta x}$$

$$+ i\ lim\Delta x \to 0 \frac{\big(v(x + \Delta x, y) - v(x, y)\big)}{\Delta x}.$$

The existence of $f'(z)$ implies that each limit in (4) exists. These limits are the definitions of the first-order partial derivatives with respect to x of u and v, respectively. Hence, we have shown two things: both ∂u/∂x and ∂v/∂x exist at the point z, and that the derivative of f is

$$f_z = \frac{\partial u}{\partial x} + \frac{i\partial v}{\partial x}$$

We now let Δz → 0 along a vertical line. With Δx = 0 and Δz = iΔy, becomes

$$f'(z) = lim\Delta y \to 0 \frac{\big(u(x, y + \Delta y) - u(x, y)\big)}{i\Delta y} + i\ lim\Delta y$$

$$\to 0 \left(\frac{v(x, y + \Delta y) - v(x, y)}{i\Delta y}\right)$$

In this case (6) shows us that ∂u/∂y and ∂v/∂y exist at z and that

$$f'(z) = -\frac{i\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

By equating the real and imaginary parts, we obtain the pair of equations.

Sufficient condition

Suppose the real functions u(x, y) and v(x, y) are continuous and have continuous first-order partial derivatives in a domain D. If u and v satisfy the Cauchy-Riemann equations (1) at all points of D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.

Sufficient Conditions for Differentiability

If the real functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in some neighborhood of a point z, and if u and v satisfy the Cauchy-Riemann equations (1) at z, then the complex function $f(z) = u(x, y) + iv(x, y)$ is differentiable at z and $f(z)$ is given by (9).

CR equations in polar coordinates

We saw that a complex function can be expressed in terms of polar coordinates. Indeed, the form $f(z) = u(r, \theta) + iv(r, \theta)$ is often more convenient to use. In polar coordinates the Cauchy-Riemann equations become

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

The polar version of CR equation at a point z whose polar coordinates are (r, θ) is then

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{1}{r}\, e^{-i\theta}\left(\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right).$$

Theorem (Complex form of the Cauchy-Riemann Equations). Suppose the formula for $f(z)$ involves z and $\overline{z}$. We can view $f(z)$ as a function of z and $\overline{z}$ and write: $f(z) = g(z, \overline{z})$.

The complex form of the Cauchy-Riemann equations is $\frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial \bar{z}} = 0$.

Assignment

1. Suppose $u(x, y)$ and $v(x, y)$ are the real and imaginary parts of an analytic function f. Can $g(z) = v(x, y) + iu(x, y)$ be an analytic function? Discuss and defend your answer with sound mathematics.
2. Suppose $f(z)$ is analytic. Can $g(z) = \overline{f(z)}$ be analytic? Discuss and defend your answer with sound mathematics.
3. If $f(z)$ and $\overline{f\ (z)}$ are both analytic in a domain D, then what can be said about f throughout D?
4. Consider the function

$$f(z) = \begin{cases} 0 & z = 0 \\ \dfrac{z^5}{|z^4|} & z \neq 0 \end{cases}$$

(**a**) Express f in the form $f(z) = u(x, y) + iv(x, y)$,

(**b**) Show that f is not differentiable at the origin.

(**c**) Show that the Cauchy-Riemann equations are satisfied at the origin. [Hint: Use the limit definitions ofthe partial derivatives ∂u/∂x, ∂u/∂y, ∂v/∂x, and ∂v/∂y at (0, 0).]

Lecture 8. Harmonic functions and harmonic conjugate

The second-order partial differential equation

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2} = 0.$$

This equation, one of the most famous in applied mathematics, is known as **Laplace's equation** in two variables. The sum $\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}$ of the two second partial derivatives is denoted by $\nabla^2\varphi$ and is called the **Laplacian** of φ. Laplace's equation is then abbreviated as $\nabla^2\varphi = 0$.

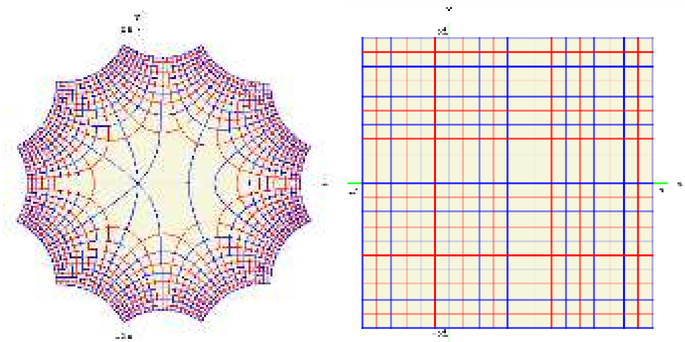
A solution $\varphi(x,y)$ of Laplace's equation in a domain D of the plane is given a special name.

A real-valued function φ of two real variables x and y that has continuous first and second-order partial derivatives in a domain D and satisfies Laplace's equation is said to be **harmonic** in D.

Suppose the complex function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D. Then the functions $u(x,y)$ and $v(x,y)$ are harmonic in D.

We have just shown that if a function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then its real and imaginary parts u and v are necessarily harmonic in D. Now suppose u(x, y) is a given real function that is known to be harmonic in D. If it is possible to find another real harmonic function v(x, y) so that u and v satisfy the Cauchy-Riemann equations throughout the domain D, then the function v(x, y) is called a **harmonic conjugate** of u(x, y). By combining the functions as u(x, y) + iv(x, y) we obtain a function that is analytic in D.

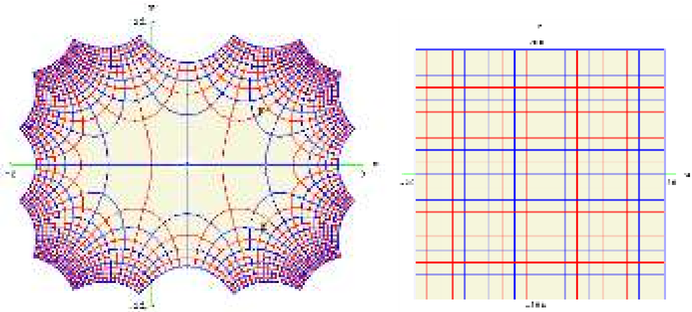
Example: Construct the harmonic conjugate of $u(x, y) = x^3 - 3xy^2 - x^2 + y^2 + 2$. Also, show that the underlying analytic function is $w = f(z) = z^3 - z^2 + 2$.



The orthogonal grid in the z-plane and it's image under the analytic function $w = f(z) = z^3 - z^2 + 2$.

Example: Construct the harmonic conjugate of $u(x, y) = x^4 + y^4 - 6x^2y^2 - 2x^3 + 6xy^2 - x^2 + y^2 + 2x + 10$.

Also, show that the underlying analytic function is $w = f(z) = z^4 - 2z^3 - z^2 + 2z + 10$.



The orthogonal grid in the z-plane and it's image under the analytic function $w = f(z) = z^4 - 2z^3 - z^2 + 2z + 10$.

EXAMPLE Harmonic Conjugate

(**a**) Verify that the function $u(x, y) = x^3 - 3xy^2 - 5y$ is harmonic in the entire complex plane.

(**b**) Find the harmonic conjugate function of u.

Example: $u(x, y) = xy + x + 2y;\ f(2i) = -1 + 5i$

Example: $u(x, y) = 4xy^3 - 4x^3y + x;\ f(1 + i) = 5 + 4i$

Example: Show that $v(x, y) = x/(x^2 + y^2)$ is harmonic in a domain D not containing the origin.

(**b**) Find a function $f(z) = u(x, y) + iv(x, y)$ that is analytic in domain D.

(**c**) Express the function f found in part (b) in terms of the symbol z.

Example: Suppose f(z) = u(r, θ) + iv(r, θ) is analytic in a domain D not containing the origin. Use the Cauchy-Riemann equations in the form $ru_r = v_\theta$ and $rv_r = -u_\theta$ to show that u(r, θ) satisfies Laplace's equation in polar coordinates:

$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0.$$

Assignment

If $f(z) = u(x, y) + iv(x, y)$ is an analytic function in a domain D and $f(z) \neq 0$ for all z in D, show that $\varphi(x, y) = log_e |f(z)|$ is harmonic in D.

Project:

Write detail about the applications of harmonic function and their harmonic conjugate. See for detail, **D. G. Zill, A first course in Complex analysis with applications.**

Lecture 9. Exercises of CR equations and analytic functions and harmonic functions

In Problems 1 and 2, the given function is analytic for all z. Show that the Cauchy-Riemann equations are satisfied at every point.

1. $f(z) = z^3$ **2.** $f(z) = 3z^2 + 5z - 6i$

In Problems 3–8, show that the given function is not analytic at any point.

3. $f(z) = Re(z)$ **4.** $f(z) = y + ix$

5. $f(z) = 4z - 6\bar{z} + 3$ **6.** $f(z) = \bar{z}^2$

7. $f(z) = x^2 + y^2$ **8.** $f(z) = \frac{x}{x^2+y^2} + i(\frac{y}{x^2+y^2})$

In Problems 9–16, use Theorem 3.5 to show that the given function is analytic in an appropriate domain.

9. $f(z) = e^{-x} \cos y - ie^{-x} \sin y$

10. $f(z) = x + \sin x \cosh y + i(y + \cos x \sinh y)$

11. $f(z) = e^{x^2-y^2} \cos 2xy + ie^{x^2-y^2} \sin 2xy$

12. $f(z) = 4x^2 + 5x - 4y^2 + 9 + i(8xy + 5y - 1)$

13. $f(z) = \frac{x-1}{(x-1)^2+y^2} - \frac{iy}{(x-1)^2+y^2}$

14. $f(z) = \frac{x^3+xy^2+x}{x^2+y^2} + \frac{i(x^{2y}+y^3-y)}{x^2+y^2}$

15. $f(z) = cos\frac{\theta}{r} - isin\frac{\theta}{r}$

16. $f(z) = 5r\cos\theta + r^4\cos 4\theta + i(5r\sin\theta + r^4\sin 4\theta)$

In Problems 17 and 18, find real constants *a*, *b*, *c*, and *d* so that the given function is analytic.

17. $f(z) = 3x - y + 5 + i(ax + by - 3)$

18. $f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$

In Problems 19–22, show that the given function is not analytic at any point but is differentiable along the indicated curve(s).

19. $f(z) = x^2 + y^2 + 2ixy;\ x - axis$

20. $f(z) = 3x^2y^2 - 6ix^2y^2;\ coordinate\ axes$

21.$f(z) = x^3 + 3xy^2 - x + i(y^3 + 3x^{2y} - y);$ coordinate axes

22. $f(z) = x^2 - x + y + i(y^2 - 5y - x);\ y = x + 2$

23. Use (9) to find the derivative of the function in Problem 9.

24. Use (9) to find the derivative of the function in Problem 11.

25. We defined the complex exponential function $f(z) = e^z$ in the following manner $e^z = e^x\cos y + ie^x\sin y$.

(**a**) Show that $f(z) = e^z$ is an entire function.

(**b**) Show that $f'(z) = f(z)$

Harmonic function and harmonic conjugate

In Problems 1–8, verify that the given function u is harmonic in an appropriate domain D.

1. $u(x,y) = x$ **2.** $u(x,y) = 2x - 2xy$

3. $u(x,y) = x^2 - y^2$ **4.** $u(x,y) = x^3 - 3xy^2$

5. $u(x,y) = \log_e(x^2 + y^2)$ **6.** $u(x,y) = cos\, x\, cosh\, y$

7. $u(x,y) = e^x(x\, cos\, y - y\, sin\, y)$

8. $u(x,y) = -e^{-x}\, sin\, y$

9. For each of the functions $u(x, y)$ in Problems 1, 3, 5, and 7, find $v(x, y)$, the harmonic conjugate of u. Form the corresponding analytic function $f(z) = u + iv$.

10. Repeat Problem 9 for each of the functions $u(x, y)$ in Problems 2, 4, 6, and 8.

Lecture 10. Linear Transformation and linear mapping

We define a **complex linear function** to be a function of the form $f(z) = az + \mathrm{b}$ where a and b are any complex constants.

A complex linear function

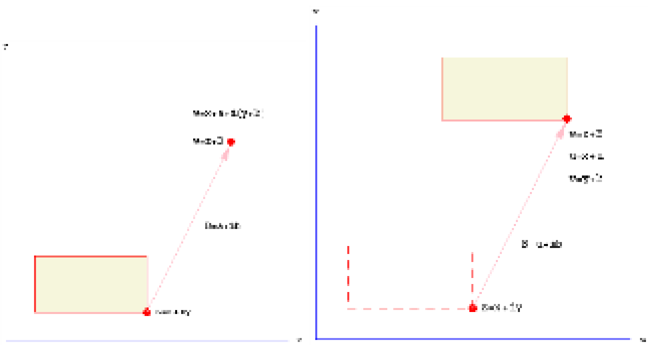
$T(z) = z + b, b \neq 0$, (1)

is called a **translation**. If we set z = x + iy and b = x0 + iy0 in (1), then we obtain:

$$T(z) = (x + iy) + (x_0 + iy_0) = x + x_0 + i(y + y_0).$$

Thus, the image of the point (x, y) under T is the point $(x + x_0, y + y_0)$.

The linear mapping $T(z) = z + b$ can be visualized in a single copy of the complex plane as the process of translating the point z along the vector (x_0, y_0) to the point T(z).



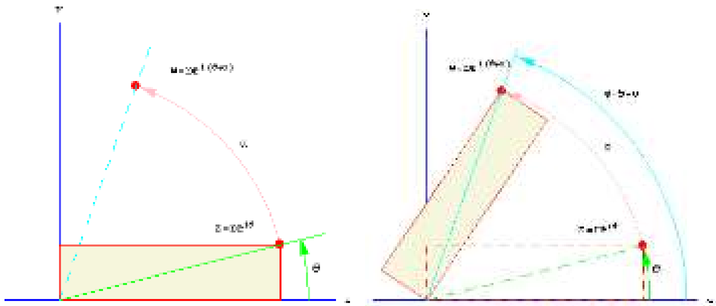
A complex linear function

$$R(z) = az, |a| = 1, (2)$$

is called a **rotation**. Although it may seem that the requirement |a| = 1 is a major restriction in (2), it is not. Keep in mind that the constant a in (2) is a complex constant. If α is any nonzero complex number, then a = α/ |α| is a complex number for which |a| = 1. So, for any nonzero complex number α, we have that R(z) = α
|α| z is a rotation.

Consider the rotation R given by (2) and, for the moment, assume that $Arg(a) > 0$. Since |a| = 1 and Arg(a) > 0, we can write a in exponential form as $a = e^{i\theta}$ with 0 < θ ≤ π. If we set $a = e^{i\theta}$ and $z = re^{i\alpha}$ in (2), then by property we obtain the following description of R:

$R(z) = e^{i\theta} re^{i\alpha} = re^{i(\theta+\alpha)}$.



The final type of special linear function we consider is magnification. A complex linear function

$M(z) = az, a > 0,$ (4)

is called a **magnification**. It is implicit in the inequality a > 0 that the symbol a represents a real number. Therefore, if z = x+ iy, then $M(z) = az = ax + iay$, and so the image of the point (x, y) is the point (ax, ay). Using the exponential form $z = re^{i\theta}$ of z, we can also express the function as: $M(z) = a(re^{i\theta}) = (ar)e^{i\theta}$.

Image of a Point under a Linear Mapping

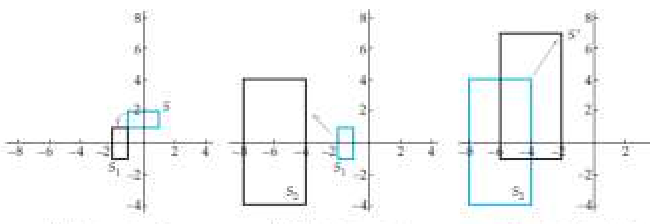
Let $f(z) = az + b$ be a linear mapping with $a \neq 0$ and let z_0 be a point in the complex plane. If the point $w_0 = f(z_0)$ is plotted in the same copy of the complex plane as z_0, then w_0 is the point obtained by

(i) rotating z_0 through an angle of Arg(a) about the origin,

(ii) magnifying the result by |a|, and

(iii) translating the result by b.

Example 1. Find the image of the rectangle with vertices −1+i, 1+i, 1+2i, and −1+2i under the linear mapping $f(z) = 4iz + 2 + 3i$.

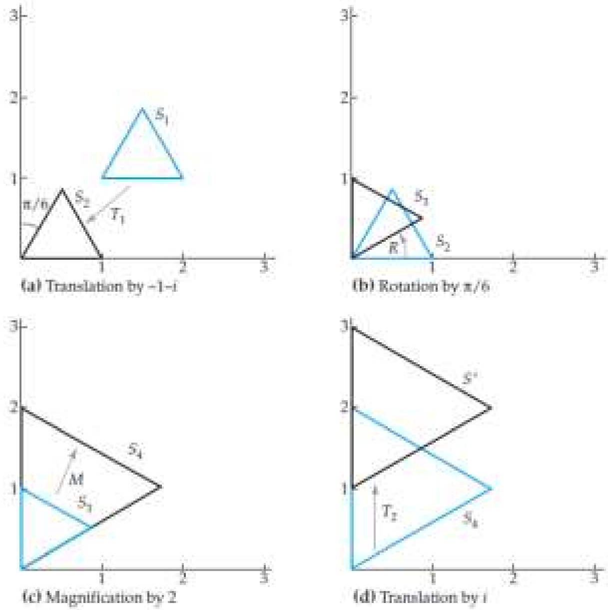


(a) Rotation by π/2 (b) Magnification by 4 (c) Translation by 2 + 3i

Example 2. Find a complex linear function that maps the equilateral triangle with vertices 1 +i, 2+i, and $\frac{3}{2} + (1 + \frac{1}{2}\sqrt{3})\, i$ onto the equilateral triangle with vertices i, $\sqrt{3} + 2i$, and 3i.

Solution Let S1 denote the triangle with vertices 1 +i, 2+i, and $\frac{3}{2} + (1 + \frac{1}{2}\sqrt{3})\, i$ shown in color in Figure (a), and let S' represent the triangle with vertices i, $\sqrt{3} + 2i$, and 3i. shown in black in Figure (d). There are many ways to find a linear mapping that maps S1 onto S'. One approach is the following: We first translate S1 to have one of its vertices at the origin. If we decide that the vertex 1 + i should be mapped onto 0, then this is accomplished by the translation $T1(z) = z - (1 + i)$. Let S2 be the image of S1 under T1. Then S2 is the triangle with vertices 0, 1, and $1/2 + 1/2\sqrt{3}i$ shown in black in Figure (a). From Figure (a), we see that the angle between the imaginary axis and the edge of S2 containing the vertices 0 and $1/2 + 1/2\sqrt{3}i$ is π/6. Thus, a rotation through an angle of π/6 radians counterclockwise about the origin will map S2 onto a triangle with two vertices on the imaginary axis. This rotation is given by $R(z) = (e^{\frac{i\pi}{6}})z = (1/2\sqrt{3} + 1/2\, i)z$, and the image of S2 under R is the triangle S3 with vertices at 0, $1/2\sqrt{3} + 1/2\, i$, and i shown in black in Figure (b). It is easy to verify that each side of the triangle S3 has length 1. Because each side of the desired triangle S_ has length 2, we next magnify S3 by a factor of 2. The magnification M(z) = 2z maps the triangle S3 shown in color in Figure (c) onto the triangle S4 with vertices $0, \sqrt{3} + i,$ and 2i shown in black in Figure (c). Finally, we translate S4 by i using the mapping T2(z) = z +i. This translation maps the triangle S4 shown in color in Figure (d) onto the triangle S_ with vertices i,√3 + 2i, and 3i shown in black in Figure (d). In conclusion, we have found that the linear mapping:

$f(z) = T2 \circ M \circ R \circ T1(z) = (\sqrt{3} + i)z + 1 - \sqrt{3} + \sqrt{3}i$ maps the triangle S1 onto the triangle S’.



(a) Translation by −1−i

(b) Rotation by π/6

(c) Magnification by 2

(d) Translation by i

Assignment:

1. Describe the action of following linear transformation on the unit disk $|z| \leq 1$

   $f(z) = 3iz + 4, f(z) = 5(cos\pi/5 + i\,sin\pi/5)z + 7i,$

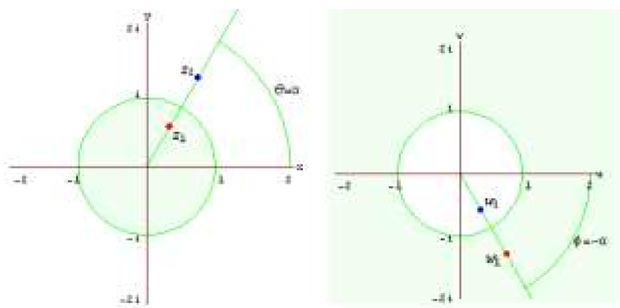
   $f(z) = -1/2z + 1 - \sqrt{3}i, f(z) = (3 - 2i)z + 12$
2. S is the triangle with vertices 0, 1, and 1 + i. S' is the triangle with vertices 2i, 3i, and −1 + 3i.
3. S is the circle $|z - 1| = 3$. S' is the circle $|z + i| = 5$.
4. S is the imaginary axis. S' is the line through the points i and 1 + 2i.
5. S is the square with vertices 1 + i, −1 + i, −1 − i, and 1 − i. S' is the square with vertices 1, 2 + i, 1+2i, and i.
6. Find two different linear mappings that map the square with vertices 0, 1, 1+i, and i, onto the square with vertices −1, 0, i, −1 + i.

Lecture 11. Mapping by $w = \frac{1}{z}$

The mapping $w = \frac{1}{z}$ is called the reciprocal transformation and maps the z-plane one-to-one and onto the w-plane except for the point z=0, which has no image, and the point w=0, which has no pre image or inverse image. Use the exponential notation $w = \rho e^{i\phi}$ in the w-plane. If $z = re^{i\theta} \neq 0$, we have

$$w = \rho e^{i\phi} = \frac{1}{z} = \frac{1}{r} e^{-i\theta}.$$

The geometric description of the reciprocal transformation is now evident. It is an inversion (that is, the modulus of (1/z) is the reciprocal of the modulus of z) followed by a reflection through the x axis. The ray $r > 0, \theta = \alpha$, is mapped one-to-one and onto the ray $\rho > 0, \phi = -\alpha$. Points that lie inside the unit circle $C_1(0) = \{z : |z| < 1\}$ are mapped onto points that lie outside the unit circle and vice versa. The situation is illustrated in Figure



Example. Show that the image of the right half plane $A = \left\{ z : \operatorname{Re}(z) \geq \frac{1}{2} \right\}$ under the mapping $w = f(z) = \frac{1}{z}$ is the closed disk $\overline{D_1(1)} = \{ w : |w - 1| \leq 1 \}$ in the w-plane.

Solution. We get the inverse mapping of $u + iv = w = f(z) = \frac{1}{z}$ as $z = f^{-1}(w) = \frac{1}{w}$. Then

$$u + iv = w = f(z) \in \overline{D_1(1)}$$

$$\Leftrightarrow f^{-1}(w) = x + iy \in A$$

$$\Leftrightarrow \frac{1}{u + iv} = x + iy \in A$$

$$\Leftrightarrow \frac{u}{u^2 + v^2} + i \frac{-v}{u^2 + v^2} = x + iy \in A$$

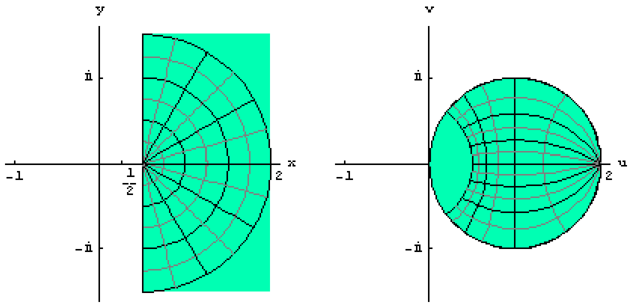
$$\Leftrightarrow \frac{u}{u^2 + v^2} = x \geq \frac{1}{2}, \quad \text{and} \quad \frac{-v}{u^2 + v^2} = y$$

$$\Leftrightarrow \frac{u}{u^2 + v^2} \geq \frac{1}{2}$$

$$\Leftrightarrow u^2 - 2u + 1 + v^2 \leq 1$$

$$\Leftrightarrow (u - 1)^2 + v^2 \leq 1$$

which describes the disk $\overline{D_1(1)}$. As the reciprocal transformation is one-to-one, pre images of the points in the disk $\overline{D_1(1)}$ will lie in the right half-plane $\operatorname{Re}(z) \geq \frac{1}{2}$. Figure 2.23 illustrates this result.



Example. For the transformation $w = f(z) = \frac{1}{z}$, find the image of the portion of the right half plane $\operatorname{Re}(z) > \frac{1}{2}$ that lies inside the closed disk $\overline{D_1\left(\frac{1}{2}\right)} = \left\{z : \left| z - \frac{1}{2} \right| \leq 1\right\}$.

Solution. We need only find the image of the closed disk $\overline{D_1\left(\frac{1}{2}\right)}$ and intersect it with the closed disk $\overline{D_1(1)}$. To begin, we note that

$$\overline{D_1\left(\frac{1}{2}\right)} = \left\{(x, y) : x^2 + y^2 - x \leq \frac{3}{4}\right\}.$$

Because $z = f^{-1}(w) = \frac{1}{w}$, we have, as before,

$$u + i v = w \in f\left(\overline{D_1\left(\frac{1}{2}\right)}\right)$$

$$\Leftrightarrow f^{-1}(w) = \frac{1}{w} \in \overline{D_1\left(\frac{1}{2}\right)}$$

$$\Leftrightarrow \frac{1}{u + i v} \in \overline{D_1\left(\frac{1}{2}\right)}$$

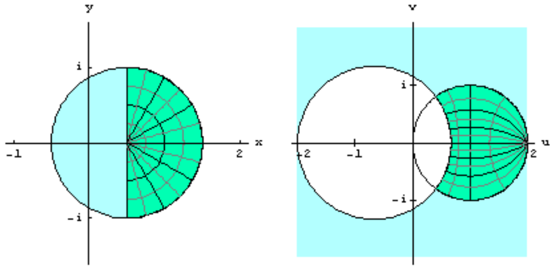
$$\Leftrightarrow \frac{u}{u^2 + v^2} + i \frac{-v}{u^2 + v^2} = x + i y \in \overline{D_1\left(\frac{1}{2}\right)}$$

$$\Leftrightarrow \left(\frac{u}{u^2 + v^2}\right)^2 + \left(\frac{-v}{u^2 + v^2}\right)^2 - \frac{u}{u^2 + v^2} \leq \frac{3}{4}$$

$$\Leftrightarrow \frac{1}{u^2 + v^2} - \frac{u}{u^2 + v^2} \leq \frac{3}{4}$$

$$\Leftrightarrow \left(u + \frac{2}{3}\right)^2 + v^2 \geq \left(\frac{4}{3}\right)^2$$

which is an inequality that determines the set of points in the w plane that lie on and outside the circle $C_{\frac{4}{3}}\left(-\frac{2}{3}\right) = \left\{w : \left|w + \frac{2}{3}\right| = \frac{4}{3}\right\}$. Note that we do not have to deal with the point at infinity this time, as the last inequality is not satisfied when $(u, v) = (0, 0)$. When we intersect this set with $\overline{D_1(1)}$, we get the crescent-shaped region shown in Figure.



To study images of "generalized circles," we consider the equation

$$A(x^2+y^2)+Bx+Cy+D=0,$$

where A, B, C, and D are real numbers. This equation represents either a circle or a line, depending on whether $A \neq 0$ or $A = 0$, respectively. Transforming the equation to polar coordinates gives

$$Ar^2+r(B\cos\theta+C\sin\theta)+D=0.$$

Using the polar coordinate form of the reciprocal transformation, we can express the image of the curve in the preceding equation as

$$A+\rho(B\cos\phi-C\sin\phi)+D\rho^2=0,$$

which represents either a circle or a line, depending on whether $D \neq 0$ or $D = 0$, respectively. Therefore, we have shown that the reciprocal transformation $w = \frac{1}{z}$ carries the class of lines and circles onto itself.

Example. Consider the mapping $w = f(z) = \frac{1}{z}$.
(a) Find the images of the vertical lines x = a. **(b)** Find the images the horizontal lines y = b.

Solution. Taking into account the point at infinity, we see that the image of the line x=0 is the line u=0; that is, the y axis is mapped onto the v axis.

Similarly, the x axis is mapped onto the u axis. Again, the inverse mapping is $z = \frac{1}{w} = \frac{u}{u^2 + v^2} + i \frac{-v}{u^2 + v^2}$, so if $a \neq 0$, the vertical line $x = a$ is mapped onto the set of (u,v) points satisfying $\frac{u}{u^2 + v^2} = a$. For $(u, v) \neq (0,0)$, this outcome is equivalent to

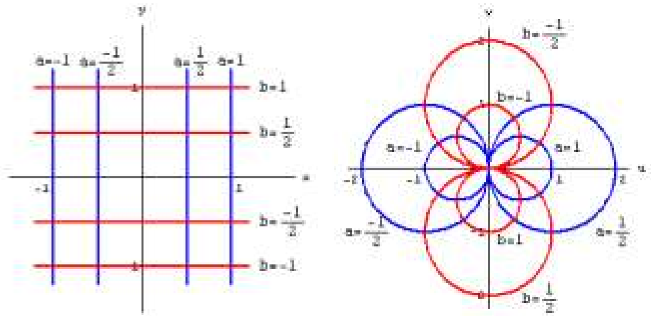
$$u^2 - \frac{1}{a} u + \frac{1}{4 a^2} + v^2 = \left(u - \frac{1}{2 a}\right)^2 + v^2 = \left(\frac{1}{2 a}\right)^2,$$

which is the equation of a circle in the w plane with center $w_0 = \frac{1}{2 a}$ and radius $\left|\frac{1}{2 a}\right|$. The point at infinity is mapped to $(u, v) = (0,0)$.

Similarly, the horizontal line $y = b$ is mapped onto the circle

$$u^2 + v^2 + \frac{1}{b} v + \frac{1}{4b^2} = u^2 + \left(v + \frac{1}{2b}\right)^2 = \left(\frac{1}{2b}\right)^2$$

which has center $w_0 = -\frac{i}{2b}$ and radius $\left|\frac{1}{2b}\right|$. Figure illustrates the images of several lines.



Assignment:

Find the image of the given set under the reciprocal mapping $w = \frac{1}{z}$ on the extended complex plane.

1. the circle $|z| = 5$

2. the semicircle $|z| = 1/2, \pi/2 \leq arg(z) \leq 3\pi/2$

3. the semicircle $|z| = 3, -\pi/4 \leq arg(z) \leq 3\pi/4$

4. the quarter circle $|z| = 1/4 , \pi/2 \leq arg(z) \leq \pi$

5. the annulus $1/3 \leq |z| \leq 2$

6. the region $1 \leq |z| \leq 4, 0 \leq arg(z) \leq 2\pi/3$

7. the ray $arg(z) = \pi/4$

8. the line segment from −1 to 1 on the real axis excluding the point z = 0

9. the line y = 4

10. the line x = 1/6

Find the image of the given set under the reciprocal mapping w = 1/z on the extended complex plane.

11. the circle |z + i| = 1

12. the circle |z + 1/3 i|= 1/3

13. the circle |z − 2| = 2

14. the circle |z + ¼|= ¼

Projects:

1. Show that the image of the line $x = k, x \neq 0$, under the reciprocal map defined on the extended complex plane is the circle $|w - \frac{1}{2k}| = |\frac{1}{2k}|$

2. If A, B, C, and D are real numbers, then the set of points in the plane satisfying the equation: $A(x^2 + y^2) + Bx + Cy + D = 0$ is called a **generalized circle**.

(**a**) Show that if A = 0, then the generalized circle is a line.

(**b**) Suppose that $A \neq 0$ and let $\Delta = B^2 + C^2 - 4AD$. Complete the square in x and y to show that a generalized circle is a circle centered at $\left(-\frac{B}{2A}, -\frac{C}{2A}\right)$ with radius $\sqrt{\frac{\Delta}{2A}}$ provided $\Delta > 0$. (If $\Delta < 0$, the generalized circle is often called an imaginary circle.)

3. Consider the complex function $f(z) = \frac{1+i}{z} + 2$ defined on the annulus $1 \leq |z| \leq 2$.

(**a**) Use mappings to determine upper and lower bounds on the modulus of $f(z) = \frac{1+i}{z} + 2$. That is, find real values L and M such that $L \leq |(1 + i)/z + 2| \leq M$.

(**b**) Find values of z that attain your bounds in (a). In other words, find z0 and z1 such that z0 and z1 are in the annulus $1 \leq |z| \leq 2$ and $|f(z0)| = L$ and $|f(z1)| = M$.

Lecture 12. Mapping of $w = z^n$ and $w = z^{\frac{1}{n}}$

Mapping of $w = z^n$, for $n \geq 2$.

The function z^2: Values of the complex power function $f(z) = z^2$ are easily found using complex multiplication. We begin by expressing this mapping in exponential notation by replacing the symbol z with $e^{i\theta}$:

$w = z^2 = \left(re^{i\theta}\right)^2 = r^2e^{i2\theta}$ (1)

From (1) we see that the modulus r^2 of the point w is the square of the modulus r of the point z, and that the argument 2θ of w is twice the argument θ of z. If we plot both z and w in the same copy of the complex plane, then w is obtained by magnifying z by a factor of r and then by rotating the result through the angle θ about the origin.

It is important to note that the magnification or contraction factor and the rotation angle associated to $f(z) = z^2$ depend on where the point z is located in the complex pla For example, since (2) $=$ $4\ and\ f(i/2) = -1/4$, the point $z = 2$ is magnified by 2 but not rotated, whereas the point $z = \frac{i}{2}$ is contracted by 1/2 and rotated through $\frac{\pi}{4}$.

Example 1. Find the image of the circular arc defined by $|z| = 2$, $0 \le \arg(z) \le \pi/2$, under the mapping $w = z^2$.

Solution Let C be the circular arc defined by $|z| = 2$, $0 \le \arg(z) \le \pi/2$, shown in Figure (a), and let C' denote the image of C under $w = z^2$. Since each point in C has modulus 2 and since the mapping $w = z^2$ squares the modulus of a point, it follows that each point in C' has modulus $2^2 = 4$. This implies that the image C' must be contained in the circle $|w| = 4$ centered at the origin with radius 4. Since the arguments of the points in C take on every value in the interval $[0, \pi/2]$ and since the mapping $w = z^2$ doubles the argument of a point, it follows that the points in C' have arguments that take on every value in the interval $[2.0, 2(\pi/2)] = [0, \pi]$. That is, the set C' is the semicircle defined by $|w| = 4, 0 \le arg(w) \le \pi$. In conclusion, we have shown that $w = z^2$ maps the circular arc C onto the semicircle C' shown.

Example 2. Find the image of the vertical line $x = k$ under the mapping $w = z^2$

Solution In this example it is convenient to work with real and imaginary parts of $w = z^2$ which, are $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$, respectively. Since the vertical line $x = k$ consists of the points $z = k + iy, -\infty < y < \infty$, it follows that the image of this line consists of all points $w = u + iv$ where $u = k^2 - y^2, v = 2ky, -\infty < y < \infty$.

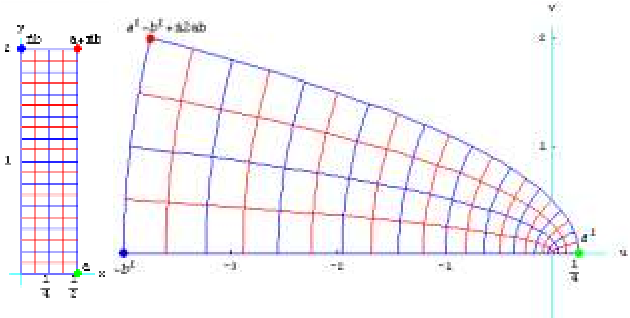
If $k \neq 0$, then we can eliminate the variable y by solving the second equation for $y = \frac{v}{2k}$ and then substituting this expression into the remaining equation and inequality. After simplification, this yields:

$u = k^2 - \frac{v^2}{4k^2}, -\infty < v < \infty$. (3)

Thus, the image of the line x = k (with $k \neq 0$) under $w = z^2$ is the set of points in the w-plane satisfying (3). That is, the image is a parabola that opens in the direction of the negative u-axis, has its vertex at $(k^2, 0)$, and has v-intercepts at $(0, \pm 2k^2)$. Notice that the image given by (3) is unchanged if k is replaced by −k. This implies that if $k \neq 0$, then the pair of vertical lines x = k and x = −k are both mapped onto the parabola $u = k^2 - \frac{v^2}{4k^2}$ by $w = z^2$The action of the mapping $w = z^2$on vertical lines is depicted in Figure. The vertical lines $x = k, k \neq 0$, shown in color in Figure are mapped onto the parabolas shown in black in Figure 2. In particular, from (3) we have that the lines x = 3 and x = −3 shown in color in Figure are mapped onto the parabola with vertex at (9, 0) shown in black in Figure. In a similar manner, the lines $x = \pm 2$ are mapped onto the parabola with vertex at (4, 0), and the lines $x = \pm 1$ are mapped onto the parabola with vertex at (1, 0). In the case when k = 0, it follows from (2) that the image of the line x = 0 (which is the imaginary axis) is given by:$u = -y^2, v = 0, -\infty < y < \infty$.

With minor modifications, the method of Example 2 can be used to show that a horizontal line $y = k, k \neq 0$, is mapped onto the parabola $u = \frac{v^2}{4k^2} - k^2\ by\ w = z^2$. Again we see that the image in

(4) is unchanged if k is replaced by −k, and so the pair of horizontal lines y = k and $y = -k, k \neq 0$, are both mapped by $w = z^2$ onto the parabola given by(4).



Example 3. Find the image of the triangle with vertices 0, 1 + i, and 1 − i under the mapping $w = z^2$.

The function $z^n, n > 2$

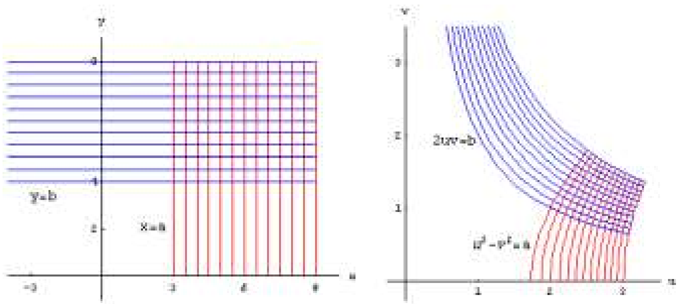
An analysis similar to that used for the mapping $w = z^{n}$ can be applied to the mapping $w = z^n, n > 2$. By replacing the symbol z with $re^{i\theta}$ we obtain: $w = z^n = r^n e^{in\theta}$. (5)

Consequently, if z and $w = z^n$ are plotted in the same copy of the complex plane, then this mapping can be visualized as the process of magnifying or contracting the modulus r of z to the modulus r^n of w, and by rotating z about the origin to increase an argument θ of z to an argument nθ of w.

The power function $z^{\frac{1}{n}}$:

The transformation $w = f(z) = z^{1/2}$ usually maps vertical and horizontal lines onto portions of hyperbolas. **(a)** Find the image of the vertical line $x = a$. **(b)** Find the image of the horizontal line $y = b$.

Solution. For $a > 0$, the algebraic equations $x = u^2 - v^2$ and $y = 2uv$ map the right half-plane given by $Re(z) > a$ (i.e., $x > a$) onto the region in the right half-plane lying to the right of the hyperbola $u^2 - v^2 = a$ and satisfying $u^2 - v^2 > a$. If $b > 0$, the equations map the upper half-plane $Im(z) > b$ (i.e., $y > b$) onto the region in the first quadrant satisfying $2uv > b$ and lying above the hyperbola $2uv = b$. This situation is illustrated in Figure. We leave as an exercise the investigation of what happens when $a = 0$ or $b = 0$.

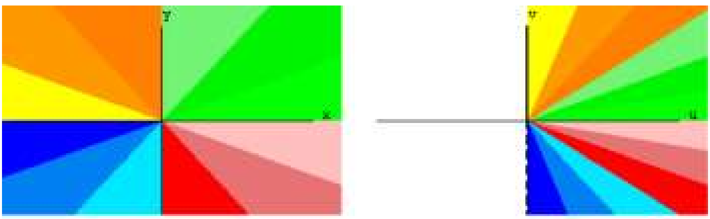


The mapping $w = f(z) = z^{1/2}$ and ($z = f^{-1}(w) = w^2$).

What happens to images of regions under the mapping $w = f(z) = z^{1/2} = |z|^{1/2} e^{i\,\mathrm{Arg}(z)/2} = r^{1/2} e^{i\theta/2}$ for $re^{i\theta} \neq 0$, where $-\pi < \theta \leq \pi$? If we use polar coordinates for $w = \rho e^{i\phi}$ in the w plane, we can represent this mapping by the system

$$\rho = r^{1/2} \text{ and } \phi = \frac{\theta}{2}.$$

This Equation indicate that the argument of f(z) is half the argument of z and that the modulus of f(z) is the square root of the modulus of z. Points that lie on the ray $r > 0,\ \theta = \alpha$ are mapped onto the ray $\rho > 0,\ \phi = \dfrac{\alpha}{2}$. The image of the z plane (with the point z=0 deleted) consists of the right half-plane Re(w)>0 together with the positive v axis. The mapping is shown in Figure.



The mapping $w = f(z) = z^{1/2}$ (and $z = f^{-1}(w) = w^2$).

Example 2. Consider the mapping $w = z^{1/2}$. Find the image of the ray $r > 0,\ \theta = \alpha$ and circle $r = c$.

We can easily extend what we've done to integer powers greater than 2. We begin by letting n be a positive integer, considering the function $w = f(z) = z^n$, for $z = re^{i\theta} \neq 0$, and then expressing it in the polar coordinate form

$$w = f(z) = z^n = r^n e^{in\theta}.$$

If we use polar coordinates for $w = \rho e^{i\phi}$ in the w -plane, this mapping can be given by the system of equations $\rho = r^n$ and

$\phi = n\theta.$

The image of the ray $r > 0$, $\theta = \alpha$ is the ray $\rho > 0$, $\phi = n\alpha$ and the angles at the origin are increased by the factor n. The functions $\cos n\theta$ and $\sin n\theta$ are periodic with period $\frac{2\pi}{n}$, so f is in general an n-to-one function; that is, n points in the z-plane are mapped onto each non-zero point in the w-plane.

If we now restrict the domain of $w = f(z) = z^n$ to the region

$$E = \left\{re^{i\theta}: \ r > 0 \text{ and } \frac{-\pi}{n} < \theta \leq \frac{\pi}{n}\right\},$$

then the image of E under the mapping $w = z^n$ can be described by the set

$F = \{\rho e^{i\phi} : \rho > 0 \text{ and } -\pi < \phi \le \pi\}$, which consists of all points in the w-plane except the point w=0. The inverse mapping of f, which we shall denote by g, is then

$$z = g(w) = w^{1/n} = \rho^{1/n} e^{i\phi/n},$$

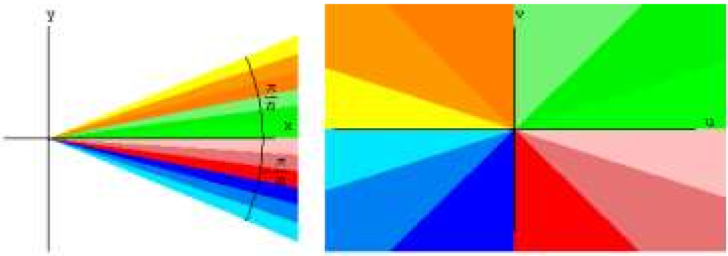
where $w \in F$. That is

$$z = g(w) = w^{1/n} = |w|^{1/n} e^{i\mathrm{Arg}(w)/n},$$

where $w \neq 0$. As with the principle square root function, we make an analogous definition for n^{th} roots.

Definition 2.2 (Principal n^{th} Root Function). The function $z = g(w) = w^{1/n} = |w|^{1/n} e^{i\mathrm{Arg}(w)/n}$, for $w \neq 0$ is called the principal n^{th} root function.

We leave as an exercise to show that f and g are inverses of each other that map the set E one-to-one and onto the set F and the set F one-to-one and onto the set E, respectively. Figure illustrates this relationship.



The mapping $w = f(z) = z^n$ (and $z = g(w) = w^{1/n}$).

Example 3. Explore the mapping $w = z^3$, $w = z^4$, $w = z^{1/3}$.

Assignment:

Find the image of the given set under the mapping $w = z^2$.

Represent the mapping by drawing the set and its image.

1. the ray $arg(z) = \pi/3$

2. the ray $arg(z) = -3\pi/4$

3. the line $x = 3$

4. the line $y = -5$

5. the line $y = -1/4$

6. the line $x = 3/2$

7. the positive imaginary axis

8. the line $y = x$

9. the circular arc $|z| = 1/2, 0 \le arg(z) \le \pi$

10. the circular arc $|z| = 4/3, -\pi/2 \le arg(z) \le \pi/6$

11. the triangle with vertices 0, 1, and 1 + i

12. the triangle with vertices 0, 1 + 2i, and −1 + 2i

13. the square with vertices 0, 1, 1 + i, and i

14. the polygon with vertices 0, 1, 1 + i, and −1 + i

Find the image of the given set under the given composition of a linear function with the squaring function.

15. the ray $arg(z) = \pi/3$; $f(z) = 2z^2 + 1 - i$

16. the line segment from 0 $to - 1 + i$; $f(z) = \sqrt{2}\, z^2 + 2 - i$

17. the line $x = 2$; $f(z) = iz^2 - 3$

18. the line $y = -3$; $f(z) = -z^2 + i$

19. the circular arc $|z| = 2, 0 \leq arg(z) \leq \frac{\pi}{2}$; $f(z) = \frac{1}{4} e^{\frac{i\pi}{4}} z^2$

20. the triangle with vertices 0, 1, and $1 + i$; $f(z) = -\frac{1}{4} iz^2 + 1$

Lecture 13. Branches of complex functions

Let $w = f(z)$ denote a function whose domain is the set D and whose range is the set R. If w is a value in the range, then there is an associated inverse function $z = g(w)$ that assigns to each value w the value (or values) of z in D for which the equation $f(z) = w$ holds. But unless f takes on the value w at most once in D, then the inverse function g is necessarily many valued, and se say that g is a multivalued function. For example, the inverse of the function $w = f(z) = z^2$ is the square root function $z = f^{-1}(w) = w^{\frac{1}{2}}$. For each value z other than $z = 0$, then, the two points z and $-z$ are mapped onto the same point $w = f(z)$; hence g is in general a two-valued function.

Let $w = f(z)$ be a multiple-valued function. A branch of f is any single-valued function f_0 that is continuous in some domain (except, perhaps, on the boundary). At each point z in the domain, assigns one of the values of $f(z)$. Associated with the branch of a function is the branch cut.

We now investigate the branches of the square root function.

Example. We consider some branches of the two-valued square root function $f(z) = z^{\frac{1}{2}}$, (where $z \neq 0$). Define the principal square root function as

$$w = f_1(z) = |z|^{\frac{1}{2}} e^{i\frac{\mathrm{Arg}(z)}{2}} = r^{\frac{1}{2}} e^{i\frac{\theta}{2}}$$
$$= r^{\frac{1}{2}} \left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right),$$

where $r = |z|^{\frac{1}{2}}$ and $\theta = \mathrm{Arg}(z)$ so that $-\pi < \theta \leq \pi$. The function $f_1(z)$ is a branch of $f(z)$. Using the same notation, we can find other branches of the square root function. For example, if we let

$$w = f_2(z) = |z|^{\frac{1}{2}} e^{i\frac{\mathrm{Arg}(z)+2\pi}{2}} = r^{\frac{1}{2}} e^{i\frac{\theta+2\pi}{2}}$$
$$= r^{1/2}\left(\cos\frac{\theta + 2\pi}{2} + i\sin\frac{\theta + 2\pi}{2}\right),$$

then

$$\begin{aligned} f_2(z) &= r^{\frac{1}{2}} e^{i\frac{\theta+2\pi}{2}} \\ &= r^{1/2} e^{i\frac{\theta}{2}} e^{i\pi} \\ &= -r^{1/2} e^{i\frac{\theta}{2}} \\ &= -f_1(z) \end{aligned}$$

so $f_1(z)$ and $f_2(z)$ can be thought of as "plus" and "minus" square root functions. The negative real axis is called a branch cut for the functions $f_1(z)$ and $f_2(z)$. Each point on the branch cut is a point of discontinuity for both functions $f_1(z)$ and $f_2(z)$.

Example. Show that the function $f_1(z) = r^{\frac{1}{2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$ is discontinuous along the negative real axis.

Solution. Let $z_0 = r_0 e^{i\pi}$ denote a negative real number. We compute the limit as z approaches z_0 through the upper half-plane $\{z : \mathrm{Im}(z) > 0\}$ and the limit as z approaches z_0 through the lower half plane $\{z : \mathrm{Im}(z) < 0\}$. In polar coordinates these limits are given by

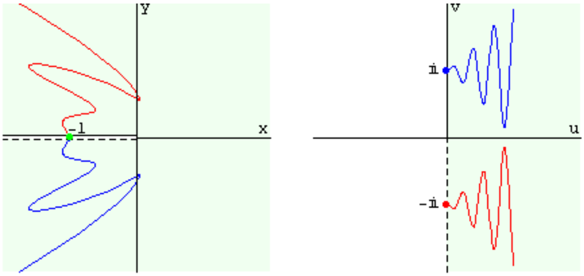
$$\lim_{(r,\theta)\to(r_0,\pi)} f_1\left(r e^{i\theta}\right) = \lim_{(r,\theta)\to(r_0,\pi)} r^{\frac{1}{2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) = i\,r_0^{1/2}$$

, and

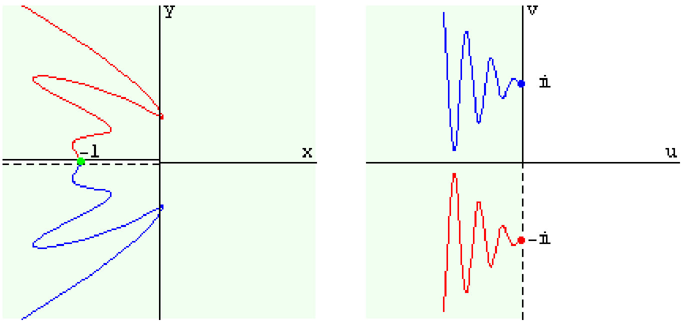
$$\lim_{(r,\theta)\to(r_0,-\pi)} f_1\left(r e^{i\theta}\right) = \lim_{(r,\theta)\to(r_0,-\pi)} r^{\frac{1}{2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) = -i\,r_0^{1/2}$$

. As the two limits are distinct, the function $f_1(z)$ is discontinuous at z_0.

Remark Likewise, $f_1(z)$ is discontinuous at z_0. The mappings $w = f_1(z)$, $w = f_2(z)$, and the branch cut are illustrated in Figure.



(a) The branch $w = f_1(z) = \sqrt{z}$ (where $z = w^2$).



(b) The branch $w = f_2(z) = -\sqrt{z}$ (where $z = w^2$).

Figure The branches $f_1(z)$ and $f_2(z)$ of $f(z) = z^{\frac{1}{2}}$.

We can construct other branches of the square root function by specifying that an argument of z given by $\theta = \arg z$ is to lie in the interval $\alpha < \theta \le \alpha + 2\pi$. The corresponding branch, denoted $f_\alpha(z)$, is $f_\alpha(z) = r^{\frac{1}{2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$,where $z = re^{i\theta} \ne 0$ and $\alpha < \theta \le \alpha + 2\pi$.

The branch cut for $f_\alpha(z)$ is the ray $r \ge 0, \theta = \alpha$, which includes the origin. The point $z = 0$, common to all branch cuts for the multivalued square root function, is called a branch point. The mapping $w = f_\alpha(z)$ and its branch cut are illustrated in Figure.

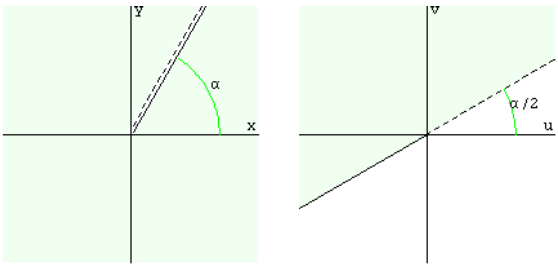
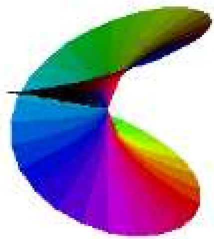


Figure The branch $f_\alpha(z)$ of $f(z) = z^{\frac{1}{2}}$.

The Riemann Surface for $w = z^{\frac{1}{2}}$

A Riemann surface is a construct useful for visualizing a multivalued function. It was introduced by Georg Friedrich Bernhard Riemann (1826-1866) in 1851. The idea is ingenious - a geometric construction that permits surfaces to be the domain or range of a multivalued function. Riemann surfaces depend on the function being investigated. We now give a nontechnical formulation of the Riemann surface for the multivalued square root function.

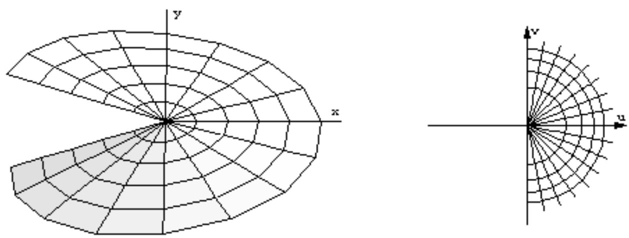


A graphical view of the Riemann surface for $w = z^{\frac{1}{2}}$.

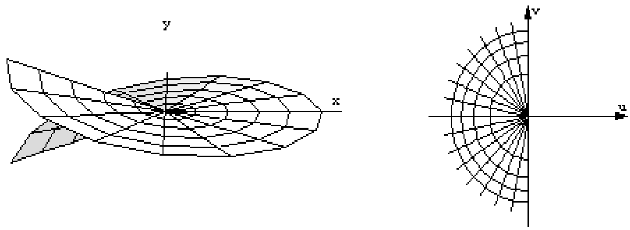
Consider $w = f(z) = z^{\frac{1}{2}}$, which has two values for any $z \neq 0$. Each function $f_1(z)$ and $f_2(z)$ in Figure 2.18 is single-valued on the domain formed by cutting the z plane along the negative x axis. Let D_1 and D_2 be the domains of $f_1(z)$ and $f_2(z)$, respectively.

The range set for $f_1(z)$ is the set H_1 consisting of the right half-plane, and the positive v axis; the range set for $f_2(z)$ is the set H_2 consisting of the left half-plane and the negative v axis. The sets H_1 and H_2 are "glued together" along the positive v axis and the negative v axis to form the w plane with the origin deleted.

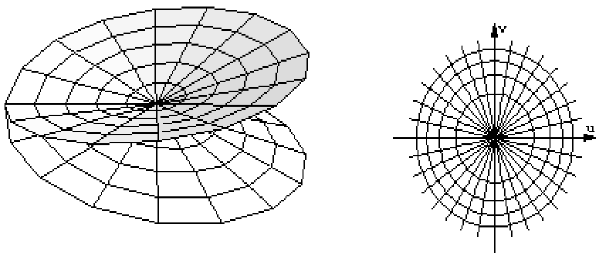
We stack D_1 directly above D_2. The edge of D_1 in the upper half-plane is joined to the edge of D_2 in the lower half-plane, and the edge of D_1 in the lower half-plane is joined to the edge of D_2 in the upper half-plane. When these domains are glued together in this manner, they form R, which is a Riemann surface domain for the mapping $w = f(z) = z^{\frac{1}{2}}$. The portions of D_1, D_2 and R that lie in $\{z : |z| < 1\}$ are shown in Figure 2.20.



(a) A portion of D_1 and its image under $w = z^{\frac{1}{2}}$.



(b) A portion of D_2 and its image under $w = z^{\frac{1}{2}}$.



(c) A portion of R and its image under $w = z^{\frac{1}{2}}$.

Formation of the Riemann surface for $w = z^{\frac{1}{2}}$.

The beauty of this structure is that it makes this "full square root function" continuous for all $z \neq 0$. Normally, the principal square root function would be discontinuous along the negative real axis, as points near -1 but above that axis would get mapped to points close to i, and points near -1 but below the axis would get mapped to points close to $-i$.

Lecture 14. Complex exponential functions and complex Logarithm

Definition 5.1 (<u>Exponential Function</u>). The definition of exp(z) is $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$. The function e^z is an entire function satisfying the following conditions:

(i). $\frac{d}{dz} e^z = e^z$, using Leibniz notation.

(ii). $e^{z_1+z_2} = e^{z_1} e^{z_2}$.

(iii). If θ is a real number, then $e^{i\theta} = cos(\theta) + i\, sin(\theta)$.

If $z = x + iy$, we also see from parts (ii) and (iii) that

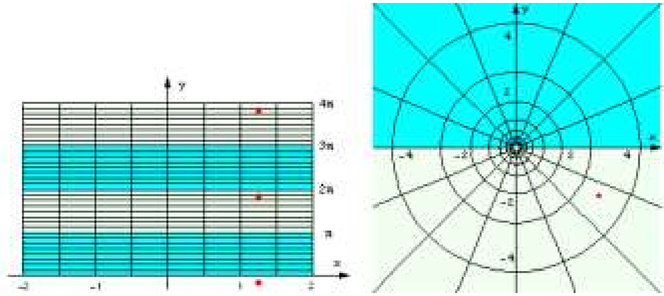
(1) $exp(z) = e^z = e^x e^{iy} = e^x(cos\ y + isin\ y)$

(2) $e^{z+i2n\pi} = e^z$, for all z, provided n is an integer,

(3) $e^z = 1$, if and only if $z = i2n\pi$, where n is an integer, and

(4) $e^{z_1} = e^{z_2}$, if and only if $z_2 = z_1 + i2n\pi$, for some integer n.

Example . For any integer n, the points $z_n = \frac{5}{4} + i(\frac{11}{6}\pi + 2n\pi)$ are mapped onto a single point in the w plane

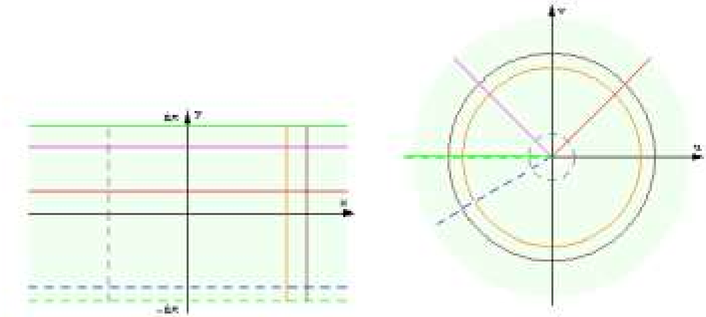


Suppose, then, that $w = e^z \neq 0$. If we write w in its exponential form as $w = \rho\, e^{\phi}$, identity gives $\rho\, e^{i\phi} = e^x e^{iy}$. we get $\rho = e^x$ and $\phi = y + 2n\pi$, where n is an integer. Therefore,

$\rho = |e^z| = e^x$, and $\phi \in arg(e^z) = \{Arg(e^z) + 2n\pi : n\ is\ an\ integer\}$.

Solving these equations for x and y, yields $x = ln\, \rho = ln\, |e^z|$ and $y = \phi + 2n\pi$, where n is an integer. Thus, for any complex number $w \neq 0$, there are infinitely many complex numbers $z = x + iy$ such that $w = e^z$. From the previous equations, we see that the numbers z are $z = x + iy = ln\, \rho + i(\phi + 2n\pi) = ln\, |w| + i(Arg\, w + 2n\pi)$, where n is an integer.

In summary, the transformation $w = e^z$ maps the complex plane (infinitely often) onto the set of nonzero complex numbers. If we restrict the solutions in equation so that only the principal value of the argument, $-\pi < Arg\, w \leq \pi$, is used, the transformation $w = e^z = e^{x+iy}$ maps the horizontal strip $\{(x, y): -\pi < y \leq \pi\}$, one-to-one and onto the range set $S = \{w : w \neq 0\}$. This strip is called the fundamental period strip.

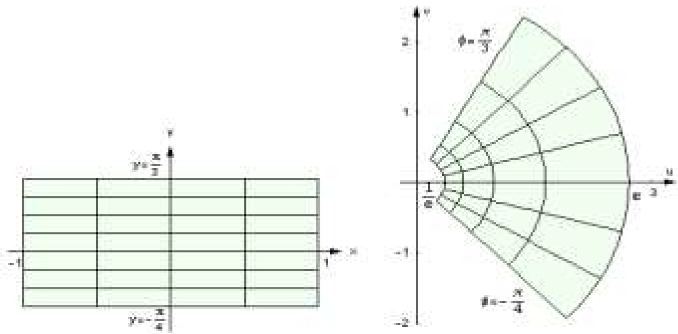


The horizontal line $z = t + i b$, for $-\infty < t < \infty$ in the z plane, is mapped onto the ray $w = e^{t} e^{i b} = e^{t} (\cos b + i \sin b)$ that is inclined at an angle $\phi = b$ in the w plane. The vertical segment $z = a + i\theta$, for $-\pi < \theta \leq \pi$ in the z plane, is mapped onto the circle centered at the origin with radius e^{a} in the w plane. That is, $w = e^{a} e^{i\theta} = e^{a} (\cos\theta + i \sin\theta)$.

Example 5.2. Consider a rectangl $R = \{(x, y) : a \leq x \leq b \text{ and } c \leq y \leq d\}$, where $-\pi < c < d \leq \pi$. Show that the transformation $w = e^{z} = e^{x+iy}$ maps the rectangle R onto a portion of an annular region bounded by two rays.

Solution. The image points in the w plane satisfy the following relationships involving the modulus and argument of w:

$e^a = \left|e^{a+iy}\right| \le \left|e^{x+iy}\right| \le \left|e^{b+iy}\right| = e^b$, and $c = \mathrm{Arg}\left(e^{x+ic}\right) \le \mathrm{Arg}\left(e^{x+iy}\right) \le \mathrm{Arg}\left(e^{x+id}\right) \le d$, which is a portion of the annulus $\left\{\rho e^{i\phi} : e^a \le \rho \le e^b\right\}$ in the w plane subtended by the rays $\phi = c$ and $\phi = d$. In Figure 5.3, we show the image of the rectangle $R^* = \left\{(x, y) : -1 \le x \le 1 \text{ and } \frac{-\pi}{4} \le y \le \frac{\pi}{3}\right\}$.



Complex Logarithm:

The multiple-valued function $\ln z$ defined by: $\ln z = \log_e |z| + i\, arg(z)$ is called the complex logarithm.

Hereafter, the notation $\ln z$ will always be used to denote the multiple valued complex logarithm. By switching to exponential notation $z = re^{i\theta}$, we obtain the following alternate description of the complex logarithm: $\ln z = \log_e |z| + i(\theta + 2n\pi), n = 0, \pm 1, \pm 2, \ldots$. We see that the complex logarithm can be used to find all solutions to the exponential equation $e^w = z$ when z is a nonzero complex number.

Example: Find all complex solutions to each of the following equations. (**a**) $e^w = i$ (**b**) $e^w = 1 + i$ (**c**) $e^w = -2$

Algebraic Properties of ln z

If z1 and z2 are nonzero complex numbers and n is an integer, then

(i) $ln(z1z2) = lnz1 + lnz2$

(ii) $ln(z1z2) =$ $z1 - ln\, z2$

(iii) $lnz1^n = n \;\; ln\, z1.$

Principal Value of the Complex Logarithm

The nplex function Ln z defined by:

$Ln\, z = log\_e|z| + iArg(z)$ is called the **principal value of the complex logarithm**.

Example: Compute the principal value of the complex logarithm $Ln\, z$ for **a**) $e^w = i$ (**b**) $e^w = 1 + i$ (**c**) $e^w = -2$

It is important to note that the identities for the complex logarithm are not necessarily satisfied by the principal value of nplex loga m. For xample, it is not true that $Ln(\; z1 \;\; z2) = Ln\, z \;\; 1 + Lnz \;\; 2$ for all complex numbers z1 and z2 (although it may be true for some complex numbers).

Ln z as an Inverse Function:

Because Ln z is one of the values of the complex logarithm ln z, it follows that:

$$e^{Lnz} = z \, for \; all \quad z \neq 0$$

This suggests that the logarithmic function Ln z is an inverse function of exponential function e^z .

Analyticity: The principal value of the complex logarithm Ln z is discontinuous at the point z= 0 since this function is not defined there. This function also turns out to be discontinuous at every point on the negative real axis. This is intuitively clear since the value of Ln z a point z near the negative x-axis in the second quadrant has imaginary part close to , whereas the value of a nearby point in the third quadrant has imaginary part close to $-\pi$. The function Ln z is, however, continuous on the set consisting of the complex plane excluding the non positive real axis. Using the theorem, a complex function $f(z) = u(xy) + iv(xy)$ is continuous at a point $z = x + iy$ if and only if both u and v are continuous real functions at (x,y). The real and imaginary parts of Ln z are $u(x, y) = log_e|z| = log_e \sqrt{x^2 + y^2}$ and $v(x, y) = Arg(z)$, respectively. From multivariable calculus we have that the function $u(x, y) = log_e \sqrt{x^2 + y^2}$ is continuous at all points in the plane except (0,0) and we have that the function v(x y) = Arg(z) is continuous on the domain $|z| > 0, -\pi < arg(z) < \pi$.

Therefore, it follows that $Ln\, z$ is a continuous function on the domain

$$|z| > 0, \quad -\pi < arg(z) < \pi$$

Put another way, the function f1 defi y:

$$f1(z) = log_e\ r + i\theta$$

is continuous on the domain where r = |z| and $\theta = arg(z)$. Since the function f1 agrees with the principal value of the complex logarithm

Ln z where they are both defined, it follows that f1 assigns to the input z one of the values of the multiple-valued function F(z) = lnz. We have shown that the function f1 defined is a branch of the multiple-valued function f(z) = lnz. (Recall that branches of a multiple-valued function F are denoted by f1, f2, and so on.) This branch is called the **principal branch of the complex logarithm**. The non positive real axis is a **branch cut** for f1 and the point z = 0 is a **branch point**. As the following theorem demonstrates, the branch f1 is an analytic function on its domain.

Theorem Analyticity of the Principal Branch of ln z

The principal branch f1 of the complex logarithm defined by is an analytic function and its derivative is given by:

$$f1(z) = 1/z$$

Logarithmic Mapping Properties

(i) w = Lnz maps the set |z| > 0 onto the region $-\infty < u < \infty$, $-\pi < v \leq \pi$.

(ii) w = Lnz maps the circle |z| = r onto the vertical line segment u = loge r,−π < v ≤ π.

(iii) w = Lnz maps the ray arg(z) = θ onto the horizontal line v = θ, −∞ < u < ∞.

Example: Find the image of the annulus 2 ≤ |z| ≤ 4 under the logarithmic mapping w =Ln z.

Assignment:

Find the image of the given set under the exponential mapping.

1. The line y = −2.

2. The line x = 3.

3. The infinite strip 1 < x ≤ 2

4. The square with vertices at 0, 1, 1 + i, and i.

5. The rectangle 0 ≤ x ≤ loge 2, −π/4 ≤ y ≤ π/2.

6. The semi-infinite strip −∞ < x ≤ 0, 0 ≤ y ≤ π.

Find a domain in which the given function f is differentiable; then find the derivative f.

7. $(z) = 3z^2 - e^{2iz} + iLn\, z$, $f(z) = (z + 1)Lnz$

8. $f(z) = \frac{Ln(2z - i)}{z^2 + 1}$

9. $f(z) = Ln(z^2 + 1)$

Find the image of the given set under the mapping w =Ln z.

10. The ray arg (z) = π/6.

12. The positive y-axis.

13. The circle |z| = 4.

14. The region in the first quadrant bounded by the circles $|z| = 1$ and $|z| = e$.

15. The annulus $3 \leq |z| \leq 5$.

Lecture 15. Complex exponents

Let c be a complex number. We define z^c as follows

$$z^c = e^{c\log(z)}.$$

The right side is a set. This definition makes sense because, if both z and c are real numbers with $z > 0$, it gives the familiar (real) definition for z^c, as the following example illustrates.

Example. Evaluate $4^{1/2}$.

Solution. Calculating $4^{\frac{1}{2}} = \exp\left[\frac{1}{2}\log 4\right]$ gives

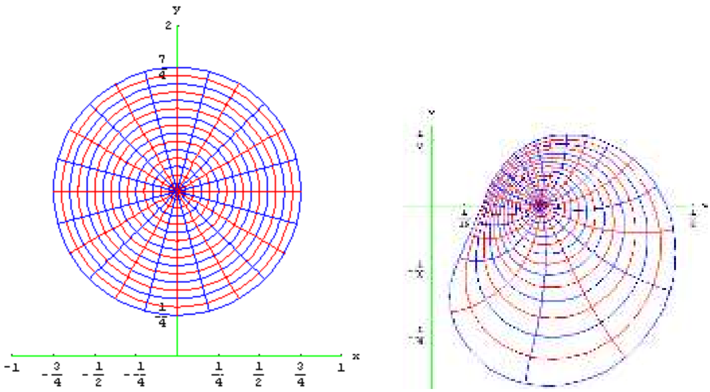
$$\frac{1}{2}\log 4 = \{\ln 2 + i n \pi : \text{ n is an integer}\}.$$

Thus $4^{\frac{1}{2}}$ is the set $\{\exp(\ln 2 + i n \pi) : \text{ n is an integer}\}$. The distinct values occur when $n = 0 \text{ and } 1$; we get $\exp(\ln 2) = 2$ and $\exp(\ln 2 + i\pi) = \exp(\ln 2)\exp(i\pi) = -2$. In other words, $4^{\frac{1}{2}} = \{-2, 2\}$.

Remark. The expression $4^{\frac{1}{2}}$ is different from $\sqrt{4}$, as the former represents the set $\{-2, 2\}$ and the latter gives only one value, $\sqrt{4} = 2$.

Because $\log(z)$ is multivalued, the function z^c will, in general, be multivalued. If we want to focus on a single value for z^c, we can do so via the function defined for $z \neq 0$ by $f(z) = \exp(c \operatorname{Log}(z))$

which is called the principal branch of the multivalued function z^c.



The mapping $w = (z)^i$.

Let us now consider the various possibilities that may arise in the definition of z^c.

Case (i). Suppose $c = k$ where k an integer. Then, if $z = r e^{i\theta} \neq 0$,

$$k \log(z) = \{k \ln(r) + i k (\theta + 2 n \pi) : \text{ n is an integer}\}.$$

Recalling that the complex exponential function has period $2\pi i$, we have

$$z^k = r^k(\cos k\theta + i\sin k\theta)\,.$$

which is the single-valued kth power of z.

Case (ii). If $c = \dfrac{1}{k}$ where k is an integer and $z = re^{i\theta} \neq 0$, then

$$\frac{1}{k}\log(z) = \left\{\frac{1}{k}\ln(r) + \frac{i(\theta + 2\pi n)}{k} : n \text{ is an integer}\right\}.$$

Hence $$z^{\frac{1}{k}} = r^{\frac{1}{k}}\left(\cos\frac{\theta + 2n\pi}{k} + i\sin\frac{\theta + 2n\pi}{k}\right)$$ for $n = 0, 1, \ldots, k-1$.

When we again use the periodicity of the complex exponential function, it gives k distinct values corresponding to $n = 0, 1, \ldots, k-1$. Therefore, as Example 5.6 illustrated, the fractional power $z^{\frac{1}{k}}$ is the multivalued k^{th} root function.

Case (iii). If j and k are positive integers that have no common factors and $c = \dfrac{j}{k}$, then becomes

$$z^{\frac{j}{k}} = r^{\frac{j}{k}}\left(\cos\frac{(\theta + 2n\pi)j}{k} + i\sin\frac{(\theta + 2n\pi)j}{k}\right)$$ for $n = 0, 1, \ldots, k-1$.

This is easy to establish. If $z = re^{i\theta}$ then

$$z^{\frac{j}{k}} = \exp\left[\frac{j}{k}\log(z)\right] = \exp\left[\frac{j}{k}\log\left(r\,e^{i\theta}\right)\right]$$

$$= \exp\left[\frac{j}{k}(\ln r + i(\theta + 2n\pi))\right]$$

$$= \exp\left[\ln r^{\frac{j}{k}} + i\frac{(\theta + 2n\pi)j}{k}\right]$$

$$= \exp\left[\ln r^{\frac{j}{k}}\right]\exp\left[i\frac{(\theta + 2n\pi)j}{k}\right]$$

$$= r^{\frac{j}{k}}\left(\cos\frac{(\theta + 2n\pi)j}{k} + i\sin\frac{(\theta + 2n\pi)j}{k}\right)$$

and again there are k distinct values corresponding to $n = 0, 1, \ldots, k - 1$.

Case (iv). Suppose c is not a rational number, then there are infinitely many values for z^c, provided $z = r\,e^{i\theta} \neq 0$.

Analyticity:

In general, the principal value of a complex power z^c defined by (6) is not a continuous function on the complex plane because the function $Ln\,z$ is not continuous on the complex plane. However, since the function e^{cz} is continuous on the entire complex plane, and since the function $Ln\,z$ is continuous on the domain $|z| > 0$, $-\pi < \arg(z) < \pi$, it follows that z^c is continuous on the domain $|z| > 0$, $-\pi < \arg(z) < \pi$. Using polar coordinates $r = |z|$ and $\theta = \arg(z)$ we have found that the function defined by:

$$f1(z) = e^{c(\log_e r + i\theta)}, -\pi < \theta < \pi$$

is a branch of the multiple-valued function $f(z) = z^c = e^{c\ln z}$.This particular branch is called the **principal branch** of the complex power z^c; its branch cut is the nonpositive real axis, and $z = 0$ is a branch point.

The bra n f agrees wit he principal value z^c on the domain |z| >0, $-\pi < arg(z) < \pi$. Consequently, the derivative of f1 can be found using the chain rule:

$$f1'(\mathrm{z}) = e^{c\ln z}\frac{C}{Z}$$

Using the principal value $z^c = e^{cLnz}$ we find that $f'1(z) = \frac{cz^c}{z} = cz^{c-1}$. That is, on the domain $|z| > 0, -\pi < arg(z) < \pi$, the principal value of the complex power z^c is differentiable and $f'(z) = \frac{cz^c}{z} = cz^{(c-1)}$.

Assignment:

Find the principal value of the given complex power.

1. $(-1)^{3i}$ **2.** $3^{\frac{2i}{\pi}}$ **3.** 2^{4i} **4.** $i^{\frac{i}{\pi}}$ **5.**$\left(1 + \sqrt{3i}\right)^{3i}$

6. $(1 + i)^{2-i}$

7. Prove that $Re\left((1+i)^{\log(1+i)}\right) = 2^{\frac{1}{4}\log(2)}e^{\frac{\pi^2}{16}}\cos\left(\frac{\pi}{4}log(2)\right)$.

Lecture 16. Trigonometric and hyperbolic functions

The complex **sine** and **cosine** functions are defined by:

$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ and $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

Analogous to real trigonometric functions, we next define the complex tangent, cotangent, secant, and cosecant functions using the complex sine and cosine:

$\tan z = \frac{\sin z}{\cos z}, \cot z = \cos\frac{z}{\sin}z, \sec z = 1/\cos z$ and $\csc z = 1/\sin z$

Most of the familiar identities for real trigonometric functions hold for the complex trigonometric functions. We now list some of the more useful of the trigonometric identities. Each of the results in is identical to its real analogue.

$$\sin(-z) = -\sin z \quad , \quad \cos(-z) = \cos z$$

$$\cos^2 z + \sin^2 z = 1$$

$$\sin(z1 \pm z2) = \sin z1 \cos z2 \pm \cos z1 \sin z2$$

$$\cos(z1 \pm z2) = \cos z1 \cos z2 \mp \sin z1 \sin z2$$

Observe that the double-angle formulas:

$$\sin 2z = 2 \sin z \cos z \quad , \cos 2z = \cos^2 z - \sin^2 z$$

Now again

$$\sin^2 z = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = \frac{e^{i2z} + e^{-2z} - 2}{-4}$$

and

$$\cos^2 z = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 = \frac{e^{i2z} + e^{-i2z} + 2}{4}$$

Adding we get $\sin^2 z + \cos^2 z = 1$.

A similar statement also holds for the complex cosine function. In summary we have:

$$\sin(z + 2\pi) = \sin z \qquad and \qquad \cos(z + 2\pi) = \cos z$$

Modulus:

If we replace the symbol z with the symbol x + iy in the expression for sin z in (4), then we obtain:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$= \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$= \sin x\left(\frac{e^{y} + e^{-y}}{2}\right) + i\cos x\left(\frac{e^{y} - e^{-y}}{2}\right)$$

Since $\cosh y = \dfrac{e^{y} + e^{-y}}{2}, \sinh y = \dfrac{e^{y} - e^{-y}}{2}$ then the above equation can be written as $\sin z = \sin x \cosh y + i \cos x \sinh z$

A similar computation enables us to express the complex cosine function in terms of its real and imaginary parts as:

$$\cos z = \cos x \cosh y + i \sin x \sinh z$$

To derive the modulus of $\sin z$ and $\cos z$ and use the identities

$$|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 z}$$
$$= \sqrt{\sin^2 x(1+\sinh^2 y) + \cos^2 x \sinh^2 z}$$
$$= \sqrt{\sin^2 x + \sinh^2 z}$$

By using similar arguments $|\cos z| = \sqrt{\cos^2 x + \sinh^2 z}$.

The Mapping $w = sin\, z$

Describe the image of the region $-\pi/2 \le x \le \pi/2, -\infty < y < \infty$, under the complex mapping $w = sinz$.

One approach to this problem ist o determine the image of vertical lines x = a with −π/2 ≤ a ≤ π/2 under w = s inz. Assume for the moment that a _= −π/2, 0, or π/2. From (16) the image of the vertical line x = a under w = s inz is given by:

$$u = sina\, cosh\, y, v = cos\, a\, sinh\, y, -\infty < y < \infty.$$

We will eliminate the variable y we obtain a single Cartesian equation relating u and v. Since −π/2 < a < π/2 and $a \neq 0$, it followsthat sin $a \neq 0$ and $cos\, a \neq 0$, and we obtain $cosh\, y = u/sin\, a$ and $sinh\, y = v/cos\, a$. The identity $cosh^2\, y - sinh^2\, y = 1$ for real hyperbolic functions then gives the following equation:

$$\left(\frac{u}{sin\, a}\right)^2 - \left(\frac{v}{cos\, a}\right)^2 = 1.$$

The Cartesian equation in (23) is a hyperbola with vertices at $(\pm sin\, a, 0)$ and slant asymptotes $v = \pm(\frac{cos\, a}{sin\, a})u$. Because the point (a, 0) is on the line x = a, the point (sin a, 0) must be on the image of the line. Therefore, the image of the vertical line x = a with −π/2 < a

< π/2 and a _= 0 under w = s in z is the branch† of the hyperbola that contains the point (sin a, 0). Because sin(−z) = −sin z for all z, it also follows that the image of the line x = −a is branch of the hyperbola containing the point (−sin a, 0).

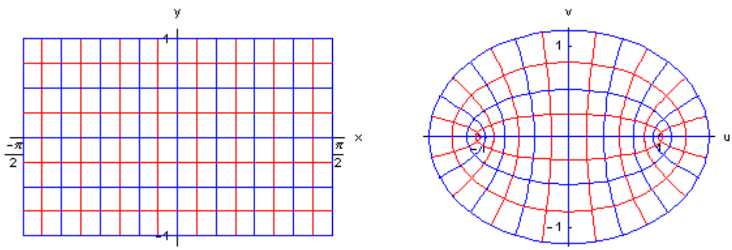
The image could also be found using horizontal line segments y = b, −π/2 ≤ x ≤ π/2, instead of vertical lines. In this case, the images are given by:

$u = s\,in x \cosh b, v = \cos x \sinh b, -\pi/2 < x < \pi/2$.

When $b \neq 0$, this set is also given by the Cartesian equation: $\left(\frac{u}{\cosh b}\right)^2 + \left(\frac{v}{\sinh b}\right)^2 = 1,$

which is an ellipse with u-intercepts at $(\pm \cosh b, 0)$ and v-intercepts at

$(0, \pm \sinh b)$. If b > 0, then the image of the line segment y = b is the upper-half of the ellipse and the image of the line segment y = −b is the bottom-half of the ellipse.



Complex hyperbolic functions

Similar arguments can be done for hyperbolic trigonometric function.

Assignment:

1. Explain the mapping of $cos\, z$.
2. Write the detail about hyperbolic functions. Follow the same lines as been done for trigonometric function.
3. Find the image of the region defined by $-\pi/2 \leq x \leq \pi/2$, $y \geq 0$, under the mapping $w = (sin\, z)^{\frac{1}{4}}$, where $z^{\frac{1}{4}}$ represents the principal fourth root function.
4. Find the period of each of the following complex functions.
   $(a) cosh\, z$ $(b) sinh\, z$ $(c)\ tanh\, z$

Lecture 17. Inverse trigonometric and hyperbolic functions

The multiple-valued function sin−1 z defined by:

$$sin^{-1} z = -i\, ln(iz + (1 - z^2)^{\frac{1}{2}})$$

iscalled the **inverse sine**.

We will also call the inverse sine the arcsine and we will denote it by $arcsin\, z$. It is clear that the inverse sine is multiple-valued since it is defined in terms of the complex logarithm $ln\, z$. It is also worth repeating that the expression $(1 - z^2)^{\frac{1}{2}}$ represents the two square roots of $(1 - z^2)$.

Example: Find all values of $sin - 1\sqrt{5}$.

Solution: By setting $z = \sqrt{5}$ in we obtain:

$$sin^{-1}\sqrt{5} = -i\, ln(i\sqrt{5} + \left(1 - \left(\sqrt{5}\right)^2\right)^{\frac{1}{2}}) =$$

$$-i\, ln(i\sqrt{5} + (-4)^{\frac{1}{2}})$$

The two square roots $(-4)^{\frac{1}{2}}$ of –4 are found to be $\pm 2i$ and so:

$$sin^{-1}\sqrt{5} = -i\, ln(i\sqrt{5} \pm 2i) = -i\, ln(\sqrt{5} \pm 2)i).$$

Because $(\sqrt{5} \pm 2)i$ is a pure imaginary number with positive imaginary part

(both $\sqrt{5} + 2$ and $\sqrt{5} - 2$ are positive), we have

$\left|(\sqrt{5} \pm 2)i\right| = \sqrt{5} \pm 2 \; and \; arg[(\sqrt{5} \pm 2)i] = \pi/2.$ Thus, we have

$$ln\left|(\sqrt{5} \pm 2)i\right| = log_e \quad \left(\sqrt{5} \pm 2\right) + i(\frac{\pi}{2} + 2n\pi)$$

For $n = 0, \pm 1, \pm 2, \ldots$. This expression can be simplified by observing that

$$log_{e}(\sqrt{5} - 2) = log_{e}(\frac{1}{\sqrt{5}+ 2}) = log_{e} 1 - log_{e}(\sqrt{5} + 2)$$
$$= 0 - log_{e}(\sqrt{5} + 2)$$

and so $loge(\sqrt{5} \pm 2) = \pm loge(\sqrt{5} + 2)$. Therefore,

$$,-iln\left|(\sqrt{5} \pm 2)i\right| = -i\left(log_e(\sqrt{5} \pm 2) + i\left(\frac{\pi}{2} + 2n\pi\right)\right) =$$

$$-i\left(\pm log_e(\sqrt{5} + 2) + i\left(\frac{\pi}{2} + 2n\pi\right)\right)$$

And so $sin^{-1}\sqrt{5} = \pm i\, log_e \quad (\sqrt{5} + 2) + \left(\frac{\pi}{2} + 2n\pi\right)$ $for\ n =$ $0, \pm 1, \pm 2, \ldots.$

Inverse Cosine and Inverse Tangent

The multiple-valued function $\cos^{-1} z$ defined by:

$$cos^{-1} z = -i\, ln[z + i(1 - z^2)^{\frac{1}{2}}]$$

Is called the **inverse cosine**. The multiple-valued function $\tan^{-1} z$ defined

by:

$$tan^{-1} z = \frac{i}{2} ln(\frac{i + z}{i - z})$$

iscalled the **inverse tangent**.

Branches and Analyticity: The inverse sine and inverse cosine are multiple-valued functions that can be made single-valued by specifying a single value of the square root to use for the expression $(1 - z^2)^{\frac{1}{2}}$ and a single value of the complex logarithm. The inverse tangent, on the other hand, can be made single-valued by just specifying a single value of $ln\ z$ to use.

A branch of a multiple-valued inverse trigonometric function may be obtained by choosing a branch of the square root function and a branch of the complex logarithm. Determining the domain of a branch defined in this manner can be quite involved. On the other hand, the derivatives of branches of the multiple-valued inverse trigonometric functions are easily found using implicit differentiation.

Inverse Hyperbolic Sine, Cosine, and Tangent

The multiple-valued functions sinh−1 z, cosh−1 z, and tanh−1 z, defined by:

$$sinh^{-1} z = ln(z + (1 + z^2)^{\frac{1}{2}})$$

$$cosh^{-1} z = ln[z + (-1 + z^2)^{\frac{1}{2}}$$

$$tanh - 1\, z = \frac{1}{2} ln(\frac{1 + z}{1 - z})$$

are called the **inverse hyperbolic sine,** the **inverse hyperbolic cosine,** and the **inverse hyperbolic tangent,** respectively.

Remarks

The multiple-valued function F(z) = sin−1 z can be visualized using the Riemann surface constructed for sin z. In order to see the image of a point z0 under the multiple-valued mapping $w = sin - 1\, z$, we imagine that z0 is lying in the xy-plane. We then consider all points on the Riemann surface lying directly over z0. Each of these points on the surface corresponds to a unique point in one of the squares Sn.. Thus, this infinite set of points in the Riemann surface represents the infinitely many images of z0 under w = sin−1 z.

Assignment:

Prove the following identities.

a) $sin^{-1}[(1 - z^2)^{\frac{1}{2}}] = cos^{-1}(\pm z)$

b) $sin^{-1} z + cos^{-1} z = \frac{1}{2}(4n + 1)\pi, n = 0, \pm 1, \pm 2, \ldots$

Lecture 18. Complex integrals

Curves Revisited: Suppose the continuous real-valued functions $x = x(t), y = y(t), a \leq t \leq b$, are parametric equations of a curve C in the complex plane. If we use these equations as the real and imaginary parts in $z = x + iy$, we can describe the points z on C by means of a complex-valued function of a real variable t called a **parametrization** of C:

$z(t) = x(t) + iy(t), a \leq t \leq b.$

The point $z(a) = x(a) + iy(a)$ or $A = (x(a), y(a))$ is called the **initial point** of C and $z(b) = x(b) + iy(b)$ or $B = (x(b), y(b))$ is its **terminal point**.

Contours: The notions of curves in the complex plane that are smooth, piecewise smooth, simple, closed, and simple closed are easily formulated in terms of the vector function (1). Suppose the derivative of (1) is $z'(t) = x'(t) + iy'(t)$. We say a curve C in the complex plane is smooth if $z'(t)$ is continuous and never zero in the interval $a \leq t \leq b$. In other words, a smooth curve has a continuously turning tangent; put yet another way, a smooth curve can have no sharp corners or cusps. A **piecewise smooth curve** C has a continuously turning tangent, except possibly at the points where the component smooth curves $C_1, C_2, \ldots, C_n$ are joined together. A curve C in the complex plane is said to be a **simple** if $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$, except possibly for t = a and t = b. C is a **closed curve** if z(a) = z(b). C is a **simple closed curve** if $z(t_1) \neq$

$z(t_2)$ for $t_1 \neq t_2$ and z(a) = z(b). In complex analysis, a piecewise smooth curve C is called a **contour** or **path**.

Complex Integral: An integral of a function f of a complex variable z that is defined on a contour C is denoted by $\int_C f(z)\,dz$ and is called a **complex integral**. Its more common name is contour integral.

Evaluation of a Contour Integral

If f is continuous on a smooth curve C given by the parameterization z(t) = x(t) + iy(t), a ≤ t ≤ b, then $\int_C f(z)dz = \int_a^b f(z(t))\, z'(t)\, dt$.

Example: Evaluate $\int_C z\,dz$, where C is given by $x = 3t, y = t^2, -1 \leq t \leq 4$.

Example: Evaluate $\oint_C \frac{1}{z}\,dz$, where C is the circle $x = \cos t, y = \sin t, 0 \leq t \leq 2\pi$.

Theorem: A Bounding Theorem

If f is continuous on a smooth curve C and if |f(z)| ≤ M for all z on C, then $\left|\int_C f(z)\,dz\right| \leq ML$, where L is the length of C.

Example: Find an upper bound for the absolute value of $\oint_C \frac{e^z}{z+1}\,dz$ where C is the Circle |z| = 4.

Assignment:

1. $\int_C (z+3)dz$ where C is $x = 2t, y = t-1, 1 \le t \le 3$
2. $\int_C (2\bar{z} - z)dz$ where C is $x = -t, y = t^2 + 2, 0 \le t \le 2$
3. $\int_C (z^2)dz$ where C is $z(t) = 3t + 2it, -2 \le t \le 2$
4. $\int_C (3z^2 + 2z)dz$ where C is $z(t) = t + it^2, 0 \le t \le 1$
5. $\int_C \frac{z+1}{z} dz$ where C is the right half of the circle |z|=1 from $z = -i$ to $z = i$
6. $\int_C |z|^2 dz$ where C is $x = t^2, y = \frac{1}{t}, 1 \le t \le 2$
7. $\oint_C Re(z)dz$ where C is the circle |z|=1.
8. $\oint_C \left(\frac{1}{(z+i)^3} - \frac{5}{z+i} + 8\right) dz$ where C is the circle |z+i|=1.
9. $\int_C (x^3 + iy^3)dz$ where C is the straight line from z=1 to z=i.
10. $\int_C (x^3 - iy^3)dz$ where C is the lower half of the circle |z|=1 from z=-1 to z=1

Cauchy Goursat's theorem and its applications

Simply and Multiply Connected Domains:

We say that a domain D is **simply connected** if every simple closed contour C lying entirely in D can be shrunk to a point without leaving D. A domain that is not simply connected is called a **multiply connected domain**; that is, a multiply connected domain has "holes" in it. We call a domain with one "hole" **doubly connected**, a domain with two "holes" **triply connected**, and so on.

Cauchy's Theorem

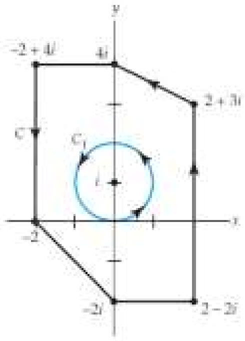
Suppose that a function f is analytic in a simply connected domain D and that f' is continuous in D. Then for every simple closed contour C in D, $\oint_C f(z)\, dz = 0.$

Proof: The proof of this theorem is an immediate consequence of Green's theorem in the plane and the Cauchy-Riemann equations.

Cauchy-Goursat Theorem for Multiply Connected Domains: If f is analytic in a multiply connected domain D with contours C and C_1 then we have $\oint_C f(z)\, dz = \oint_{C_1} f(z)\, dz.$

The last result is sometimes called the **principle of deformation of contours** since we can think of the contour C_1 as a continuous deformation of the contour C. Under this deformation of contours, the value of the integral does not change.

Example: Evaluate $\oint_C \frac{1}{z-i}\, dz$, where C is the contour



Cauchy-Goursat Theorem for Multiply Connnected Domains

Suppose $C, C1, \ldots, Cn$ are simple closed curves with a positive orientation such that $C1, C2, \ldots, Cn$ are interior to C but the regions interior to each $Ck, k = 1, 2, \ldots, n$, have no points in common. If f is analytic on each contour and at each point interior to C but exterior to all the $Ck, k = 1, 2, \ldots, n$, then

$$\oint_C f(z)\, dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\, dz.$$

Assignment:

In Problems 1–8, show that $\oint_C f(z)dz = 0$, where f is the given function and C is the unit circle $|z| = 1$.

1. $f(z) = z^3 - 1 + 3i$ **2.** $f(z) = z^2 + \frac{1}{z-4}$

3. $f(z) = \frac{z}{2z+3}$ **4.** $f(z) = \frac{z-3}{z^2+2z+2}$

5. $f(z) = \frac{\sin z}{(z^2-25)(z^2+9)}$

6. $f(z) = \frac{e^z}{2z^2 + 11z + 15}$

7. $f(z) = tan\, z$ **8.** $f(z) = \frac{z^2 - 9}{\cosh z}$

9. Evaluate $\oint_C f(z)dz, f(z) = \frac{1}{z}$, where C is the contour shown in Figure.

10. Evaluate $\oint_C f(z)dz, f(z) = \frac{5}{z+1+i}$, where C is the contour shown in Figure.

In Problems 11–22, use any of the results in this section to evaluate the given integral along the indicated closed contour(s). $f(z)$ and the contours are given.

11. $z + \frac{1}{z}$; $|z| = 2$ **12.** $z + \frac{1}{z^2}$; $|z| = 2$

13. $\frac{z}{z^2 - \pi^2}$; $|z| = 3$

14. $\frac{10}{(z + i)^4}$; $|z + i| = 1$

15. $\frac{2z + 1}{z^2 + z}$; $(\boldsymbol{a})|z| = \frac{1}{2}, (\boldsymbol{b})|z| = 2, (\boldsymbol{c})|z - 3i| = 1$

16. $\frac{2z}{z^2 + 3}$; $(\boldsymbol{a})|z| = 1, (\boldsymbol{b})|z - 2i| = 1, (\boldsymbol{c})|z| = 4$

17. $\frac{-3z + 2}{z^2 - 8z + 12}$; $(\boldsymbol{a})|z - 5| = 2, (\boldsymbol{b})|z| = 9$

18. $\frac{3}{z + 2} - \frac{1}{z - 2i}$; $(\boldsymbol{a})|z| = 5, (\boldsymbol{b})|z - 2i| = \frac{1}{2}$

19. $\frac{z - 1}{z(z - i)(z - 3i)}$; $|z - i| = \frac{1}{2}$

20. $\frac{1}{z^3 + 2iz^2}$; $|z| = 1$

21. $Ln(z + 10)$; $|z| = 2$

22. $\frac{5}{(z - 2)^3} + \frac{3}{(z - 2)^2} - \frac{10}{z - 2} + 7\, csc\, z$; $|z - 2| = \frac{1}{2}$

23. Evaluate $\frac{8z-3}{z^2-z}$, where C is the "figure-eight" contour shown in Figure. [*Hint*: Express C as the union of two closed curves $C1$ and $C2$.

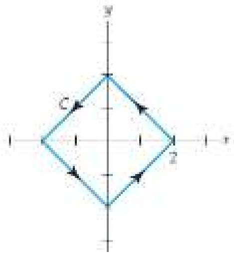


Figure 5.34 Figure for Problem 9

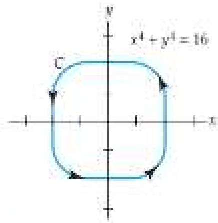


Figure 5.35 Figure for Problem 10

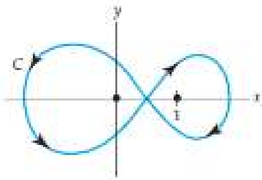


Figure 5.36 Figure for Problem 23

Lecture 19. Cauchy integral formula and their consequences

Cauchy's Integral Formula

Suppose that f is analytic in a simply connected domain D and C is any simple closed contour lying entirely within D. Then for any point z0 within C,

$$f(z0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z0}dz.$$

Proof Let D be a simply connected domain, C a simple closed contour in D, and z0 an interior point of C. In addition, let C1 be a circle centered at z0 with radius small enough so that C1 lies within the interior of C. By the principle of deformation of contours,

$$\oint_C \frac{f(z)}{z - z0}dz. = \oint_{C_1} \frac{f(z)}{z - z0}dz.$$

We wish to show that the value of the integral on the right is 2πif(z0). To this end we add and subtract the constant f(z0) in the numerator of the integrand,

$$\oint_{C_1} \frac{f(z)}{z - z0}dz = \oint_{C_1} \frac{f(z0) - f(z0) + f(z)}{z - z0}dz$$

$$= f(z0)\oint_{C_1} \frac{1}{z - z0}dz$$

$$+ \oint_{C_1} \frac{-f(z0) + f(z)}{z - z0}dz$$

Since we know that $\oint_{C_1} \frac{1}{z - z0} dz = 2\pi i$

Since f is continuous at z0, we know that for any arbitrarily small ε > 0 there exists a δ > 0 such that |f(z) − f(z0)| < ε whenever |z − z0| < δ. In particular, if we choose the circle C1 to be |z − z0| = 1
2δ < δ, then by the ML-inequality

$$\oint_{C_1} \frac{-f(z0) + f(z)}{z - z0} dz \leq 2\pi\epsilon$$

In other words, the absolute value of the integral can be made arbitrarily small by taking the radius of the circle C1 to be sufficiently small. This can happen only if the integral is 0.

$$\oint_{C_1} \frac{f(z)}{z - z0} dz = 2\pi i\, f(z0)$$

Differentiating the above equation with respect to z0 we get its extension for higher derivatives

$$\frac{n!}{2\pi i} \oint_{C_1} \frac{f(z)}{(z - z0)^{n+1}} dz = f^n(z0).$$

Some Consequences of the Integral Formulas

Cauchy's Inequality

Suppose that f is analytic in a simply connected domain D and C is a circle defined by |z − z0| = r that lies entirely in D. If |f(z)| ≤ M for all points z on C, then $|f^n\ (z0)| < \frac{n!M}{r^n}$

Proof From the hypothesis,

$$\left|\frac{f(z)}{(z - z0)^{n+1}}\right| = \frac{|f(z)|}{r^{n+1}} \leq \frac{|M|}{r^{n+1}}$$

Thus from the ML-inequality, we have

$$|f^n\,(z0)| = \frac{n!}{2\pi i}\left|\oint_{C_1} \frac{f(z)}{(z\,-\,z0)^{n+1}}\,dz\right|$$

$$= \frac{n!}{2\pi i}\oint_{C_1} \left|\frac{f(z)}{(z\,-\,z0)^{n+1}}\right| dz \le \frac{n!\,|M|}{r^{n+1}}$$

Liouville's Theorem

The only bounded entire functions are constants.

Morera's Theorem

If f is continuous in a simply connected domain D and if

$$\oint_C f(z)\,dz \;=\; 0$$

for every closed contour C in D, then f is analytic in D.

Assignment:

Evaluate the integral of the following functions along the indicated closed contour(s).

1. $\frac{4}{z-3i}$; $|z| \;=\; 5$ **2.** $\frac{z^2}{(z-3i)^2}$; $|z| = \;5$

3. $\frac{e^z}{z-\pi i}$; $|z| = \;4$ **4.** $\frac{1+e^z}{z}$; $|z| = \;1$

5. $\frac{z^2-3z+4i}{z+2i}$; $|z| = \;3$ **6.** $cos\,\frac{z}{3z-\pi}$; $|z| = \;1.1$

7. $\frac{z^2}{z^2+4}$; $(\boldsymbol{a})|z\,-\,i| = \;2,$ $(\boldsymbol{b})|z\,+\,2i| = \;1$

8. $\frac{z^2+3z+2i}{z^2+3z-4}$; $(\boldsymbol{a})|z| = \;2,$ $(\boldsymbol{b})|z\,+\,5| = \frac{3}{2}$

9. $\frac{z^2+4}{z^2-5iz-4}$; $|z\,-\,3i| = \;1.3$ **10.** $sin\,\frac{z}{z^2+\pi^2}$; $|z\,-\,2i| = \;2$

11. $\frac{e^{z^2}}{(z-i)^3}$; $|z - i| = 1$ **12.** $\frac{z}{(z+i)^4}$; $|z| = 2$

13. $\cos\frac{2z}{z^5}$; $|z| = 1$ **14.** $e^{-z}\sin\frac{z}{z^3}$; $|z - 1| = 3$

15. $\frac{2z+5}{z^2-2z}$; $(\boldsymbol{a})|z| = \frac{1}{2}$, $(\boldsymbol{b})|z + 1| = 2$ $(\boldsymbol{c})|z - 3| = 2$, $(\boldsymbol{d})|z + 2i| = 1$

16. $\frac{\boldsymbol{z}}{(z-1)(z-2)}$; $(\boldsymbol{a})|z| = \frac{1}{2}$, $(\boldsymbol{b})|z + 1| = 1$ $(\boldsymbol{c})|z - 1| = \frac{1}{2}$, $(\boldsymbol{d})|z| = 4$

17. $\frac{z+2}{z^{2(z-1-i)}}$; $(\boldsymbol{a})|z| = 1$, $(\boldsymbol{b})|z - 1 - i| = 1$

18. $\frac{1}{z^{3(z-4)}}$; $(\boldsymbol{a})|z| = 1$, $(\boldsymbol{b})|z - 2| = 1$

19. $\frac{e^{2iz}}{z^4} - \frac{z^4}{(z - i)^3}$; $|z| = 6$

20. $\cosh\frac{z}{(z-\pi)^3} - \sin 2\frac{z}{(2z-\pi)^3}$; $|z| = 3$

21. $\frac{1}{z^3(z - 1)^2}$, $|z - 2| = 5$

22. $\frac{1}{z^{2(z^2+1)}}$; $|z - i| = \frac{3}{2}$

Lecture 20. Taylor series representation and its applications

Definition (<u>Taylor Series</u>). If f(z) is analytic at $z = \alpha$, then the series

$$f(\alpha) + f'(\alpha)(z-\alpha) + f'\frac{(\alpha)(z-\alpha)^2}{2!} + \cdots = \sum \frac{f^k(\alpha)(z-\alpha)^k}{k!}$$

is called the Taylor series for f(z) centered at $z = \alpha$. When the center is $= 0$, the series is called the Maclaurin series for f(z).

Theorem (<u>Taylor's Theorem</u>). Suppose $f(z)$ is analytic in a domain G, and that $D_R(\alpha) = \{z: |z-\alpha| < R\}$ is any disk contained in G. Then the Taylor series for $f(z)$ converges to $f(z)$ for all z in $D_R(\alpha)$; that is, $f(z) = \sum \frac{f^k(\alpha)(z-\alpha)^k}{k!}$ $for\ all\ z \in D_R(\alpha)$.

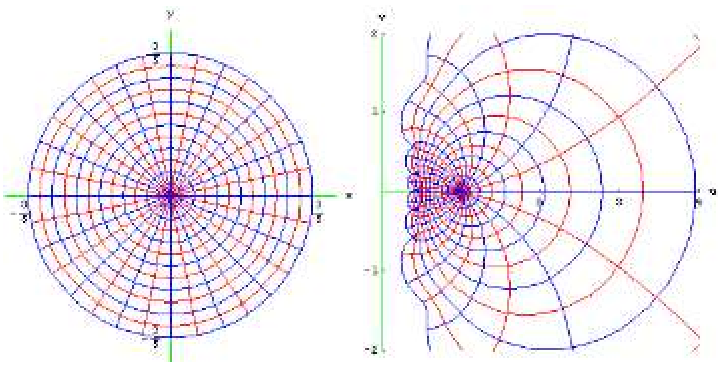
Furthermore, for any $r, 0 < r < R$, the convergence is uniform on the closed subdisk $D_r(\alpha) = \{z: |z-\alpha| < r\}$.

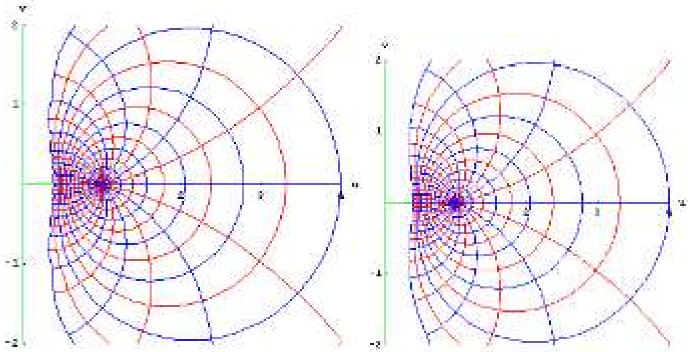
Corollary Suppose that f(z) is analytic in the domain G that contains the point $z = \alpha$. Let z_0 be a non removable singular point of minimum distance to the point $z = \alpha$. If $|z_0 - \alpha| = R$, then

(i) the Taylor series $\sum \frac{f^k(\alpha)(z-\alpha)^k}{k!}$ converges to $f(z)$ on all of $D_R(\alpha)$ (ii) if $|z_1 - \alpha| = S > R$, the series $\sum \frac{f^k(\alpha)(z_1-\alpha)^k}{k!}$ does not converges to $f(z_1)$.

Example Show that $\frac{1}{(1-z)^2} = \sum(n+1)z^n$ is valid for all $z \in D_1(0)$.

Solution: Let $f(z) = \frac{1}{(1-z)^2}$ then its nth derivative is $f^n(z) = \frac{(n+1)!}{(1-z)^{n+2}}$. the Taylor series of f(z) becomes $f(z) = \frac{1}{(1-z)^2} = \sum(n+1)z^n$.



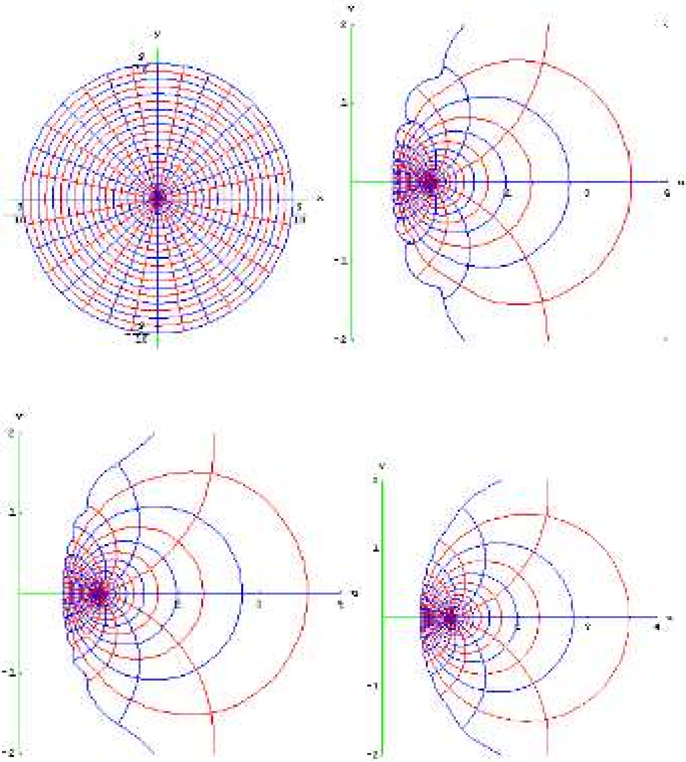


The disk $D_{\frac{3}{5}}(0)$ and its images under the finite sum $w = S_8, w = S_{12}$, and $w = S_{18}$.

Example: Show that for $z \in D_1(0)$,

a) $\frac{1}{a-z^2} = \sum \quad z^{2n}$ b) $\frac{1}{a+z^2} = \sum(-1)^n z^{2n}$

Solution: The series can be found by replacing z by z^2 *and* $-z^2$ respectively in previous example.



The disk $D_{\frac{9}{10}}(0)$ and its image under the finite sum of 12, 16 and 20 terms.

Theorem: Let f(z) and g(z) have the power series representations

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n \text{ for } z \in D_{r_1}(\alpha),$$

and

$$g(z) = \sum_{n=0}^{\infty} b_n (z - \alpha)^n \text{ for } z \in D_{r_2}(\alpha)$$
.

If $r = \min\{r_1, r_2\}$ and β is any complex constant, then

$$\beta f(z) = \sum_{n=0}^{\infty} \beta a_n (z - \alpha)^n \text{ for } z \in D_{r_1}(\alpha)$$
,

$$f(z) + g(z) = \sum_{n=0}^{\infty} (a_n + b_n)(z - \alpha)^n \text{ for } z \in D_r(\alpha)$$
, and

$$f(z) * g(z) = \sum_{n=0}^{\infty} c_n (z - \alpha)^n \text{ for } z \in D_r(\alpha)$$
, where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$
.

Identity is known as the <u>Cauchy</u> product of the series for f(z) and g(z).

Example 7.6. Use the Cauchy product of series to show that

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n \quad \text{for} \quad z \in D_1(0).$$

Solution. We let $f(z) = g(z) = \dfrac{1}{1-z}$, for $z \in D_1(0)$ we have $a_n = b_n = 1$, for all n, and thus

$$\frac{1}{(1-z)^2} = h(z) = f(z)\,g(z)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k\, b_{n-k}\right) z^n$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} 1\right) z^n$$

$$= \sum_{n=0}^{\infty} (n+1)\, z^n$$

Lecture 21. Laurant's series representation:

Definition 7.3 (<u>Laurent Series</u>). Let c_n be a complex number for $n = 0, \pm 1, \pm 2, \ldots$. The doubly infinite series $\sum \; c_n(z - \alpha)^n$, called a <u>Laurent</u> series, is defined by

$$\sum c_n(z-\alpha)^n = \sum c_{-n}(z-\alpha)^{-n} + \sum c_n(z-\alpha)^n \qquad ,$$

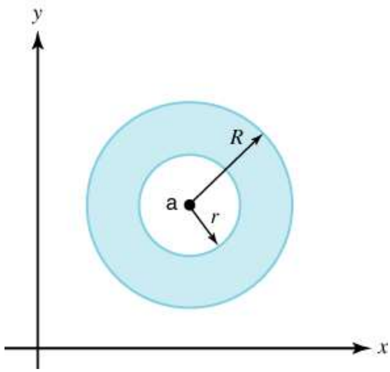
provided the series on the right-hand side of this equation converge.

Definition Given $0 \le r < R$, we define the annulus centered at α with radii r and R by

$$A(\alpha, r, R) = \{z : r < |z-\alpha| < R\}.$$

The closed annulus centered at with radii r and R is denoted by

$$\overline{A}(\alpha, r, R) = \{z : r \le |z-\alpha| \le R\}.$$



Theorem Suppose that the Laurent series $\sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n$ converges on an annulus $A(\alpha, r, R) = \{z : r < |z-\alpha| < R\}$. Then the series converges uniformly on any closed subannulus $\overline{A}(\alpha, s, t) = \{z : s \le |z-\alpha| \le t\}$ where $r < s < t < R$.

Theorem (Laurent's Theorem). Suppose $0 \le r < R$, and that f(z) is analytic in the annulus $A = A(\alpha, r, R) = \{z : r < |z-\alpha| < R\}$. If ρ is any number such that $r < \rho < R$, then for all $z_0 \in A(\alpha, r, R)$ the function value $f(z_0)$ has the Laurent series representation

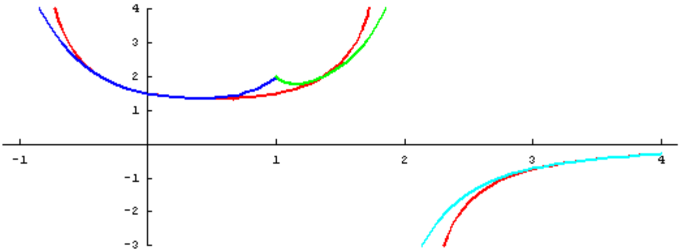
$$f(z_0) = \sum_{n=-\infty}^{\infty} c_n (z_0-\alpha)^n = \sum_{n=1}^{\infty} c_{-n} (z_0-\alpha)^{-n} + \sum_{n=0}^{\infty} c_n (z_0-\alpha)^n$$

where for $n = 0, 1, 2, \ldots$, the coefficients c_{-n} and c_n are given by

$$c_{-n} = \frac{1}{2\pi i}\int_{C_\rho^+(\alpha)} \frac{f(z)}{(z-\alpha)^{-n+1}}\,dz$$ and

$$c_n = \frac{1}{2\pi i}\int_{C_\rho^+(\alpha)} \frac{f(z)}{(z-\alpha)^{n+1}}\,dz$$

Example . Find three different Laurent series representations for the function $f(z) = \dfrac{3}{2+z-z^2}$ involving powers of z.



Solution. The function f(z) has singularities at $z=-1, 2$ and is analytic in the disk $D: |z|<1$, in the annulus $A: 1<|z|<2$, and in the region $R: |z|>2$. We want to find a different Laurent series for f(z) in each of the three domains D, A, and R. We start by writing f(z) in its partial fraction form:

$$f(z) = \frac{3}{(1+z)(2-z)} = \frac{1}{1+z} + \frac{1}{2-z} = \frac{1}{1+z} + \frac{1}{2}\frac{1}{1-\frac{z}{2}}$$

We use Theorem 4.12 and Corollary 4.1 to obtain the following representations for the terms on the right side

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \qquad \text{valid for } |z|<1,$$

$$\frac{1}{1+z} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n} \qquad \text{valid for } |z|>1,$$

$$\frac{1}{2}\frac{1}{1-\frac{z}{2}} = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \qquad \text{valid for } |z|<2,$$

$$\frac{1}{2}\,\frac{1}{1-\frac{z}{2}} = \sum_{n=1}^{\infty} \frac{-2^{n-1}}{z^{n}} \qquad \text{valid for } |z| > 2 .$$

thus we have

$$f(z) = \sum_{n=0}^{\infty} (-1)^{n} z^{n} + \sum_{n=0}^{\infty} \frac{z^{n}}{2^{n+1}} = \sum_{n=0}^{\infty} \left((-1)^{n} + \frac{1}{2^{n+1}} \right) z^{n}$$

valid for $|z| < 1$, which is a Laurent series that reduces to a Maclaurin series.

In the annulus $A: 1 < |z| < 2$, we get

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^{n}} + \sum_{n=0}^{\infty} \frac{z^{n}}{2^{n+1}}$$

valid for $1 < |z| < 2$.

Finally, in the region $R: |z| > 2$ we obtain

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^{n}} + \sum_{n=1}^{\infty} \frac{-2^{n-1}}{z^{n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} - 2^{n-1}}{z^{n}}$$

valid for $|z| > 2$.

Lecture 22. Singularities, zeros and poles

Recall that the point $z = \alpha$ is called a singular point, or singularity of the complex function f(z) if f is not analytic at $z = \alpha$, but every neighborhood $D_R(\alpha)$ of α contains at least one point at which f(z) is analytic.

The point α is called a isolated singularity of the complex function f(z) if f is not analytic at $z = \alpha$, but there exists a real number $R > 0$ such that f(z) is analytic everywhere in the punctured disk $D_R^*(\alpha)$. The function $f(z) = \dfrac{1}{1 - z}$ has an isolated singularity at $z = 1$.

Definition (Removable Singularity, Pole of order k, Essential Singularity). Let f(z) have an isolated singularity at α with Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - \alpha)^n \quad \text{valid for} \quad z \in A(\alpha, 0, R).$$

Then we distinguish the following types of singularities at α.

(i) If $c_n = 0$ for $n = -1, -2, -3, \ldots$, then we say that f(z) has a removable singularity at α.

(ii) If k is a positive integer such that $c_{-k} \neq 0$ but $c_n = 0$ for $n = -k-1, -k-2, -k-3, \ldots$, then we say that f(z) has a pole of order k at α.

(iii) If $c_n \neq 0$ for infinitely many negative integers n, then we say that f(z) has an essential singularity at $z = \alpha$.

Discuss the examples $\frac{\sin z}{z}, \frac{\cos z - 1}{z^2}, \sin\frac{z}{z^2}, \frac{e^z}{z}, z^2 \sin(\frac{1}{z}),$

Theorem A function f(z) analytic in the punctured disk $D_R^*(\alpha)$ has a pole of order k at $z = \alpha$ if and only if it can be expressed in the form

$$f(z) = \frac{h(z)}{(z-\alpha)^k},$$

where the function h(z) is analytic at the point $z = \alpha$ and $h(\alpha) \neq 0$.

Assignment:

In the following problems, determine the order of the poles for the given function.

1. $f(z) = \frac{3z - 1}{z^2 + 2z + 5}$ **2.** $f(z) = 5 - \frac{6}{z^2}$

3. $f(z) = \frac{1 + 4i}{(z + 2)(z + i)^4}$ **4.** $f(z) = \frac{z - 1}{(z + 1)(z^3 + 1)}$

5. $f(z) = tan\, z$ **6.** $f(z) = \frac{\cot \pi z}{z^2}$

7. $f(z) = \frac{1 - cosh\, z}{z^4}$ **8.** $f(z) = \frac{e^z}{z^2}$

9. $f(z) = \frac{1}{1 + e^z}$ **10.** $f(z) = \frac{e^z - 1}{z^2}$

11. $f(z) = \frac{\sin z}{z^2 - z}$ **12.** $f(z) = \frac{cos\, z - cos\, 2z}{z^6}$

Lecture 23. Residue theorem

We saw in the last section that if a complex function f has an isolated singularity at a point z0, then f has a Laurent series representation

$$f(z) = \sum a_k(z - z0)^k$$

$$= \cdot\ \cdot\ \cdot + \frac{a_{-2}}{(z - z0)^2} + \frac{a_{-1}}{z - z0}$$

$$+ a0 + a_1(z - z0) + \cdot\ \cdot\ \cdot\ ,$$

which converges for all z near z0. More precisely, the representation is valid in some deleted neighborhood of z0 or punctured open disk $0 < |z - z0| < R$. In this section our entire focus will be on the coefficient a_{-1} and its importance in the evaluation of contour integrals.

Residue The coefficient a_{-1} of $\frac{1}{z - z0}$ in the Laurent series given above is called the **residue** of the function f at the isolated singularity z0. We shall use the notation $a_{-1} = Res(f(z), z0)$ to denote the residue of f at z0.

Residue at a Simple Pole

If f has a simple pole at z = z0, then $Res(f(z), z0) = \lim_{z \to z0}(z - z0)f(z)$.

Residue at a Pole of Order n

If f has a pole of order n at z = z0, then

$$Res(f(z), z0) = \frac{1}{(n-1)!} \lim_{z \to z} \frac{d^{n-1}}{dz^{n-1}} (z - z0)^n f(z)\ .$$

Example: Residue at a Pole

The function $f(z) = \frac{1}{(z-1)^2(z-3)}$ has a simple pole at $z = 3$ and a pole of order 2 at $z = 1$.

Cauchy's Residue Theorem

Let D be a simply connected domain and C a simple closed contour lying entirely within D. If a function f is analytic on and within C, except at a finite number of isolated singular points $z1, z2, \dots, zn$ within C, then

$$\oint_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} Res\,(f(z), zk).$$

Assignment:

Find the residue at each pole of the given function.

1. $f(z) = \frac{z}{z^2} + 16$

2. $f(z) = \frac{4z+8}{2z-1}$

3. $f(z) = \frac{1}{z^4 + z^3 - 2z^2}$

4. $f(z) = \frac{1}{(z^2 - 2z + 2)^2}$

5. $f(z) = \frac{5z^2 - 4z + 3}{(z+1)(z+2)(z+3)}$

6. $f(z) = \frac{2z-1}{(z-1)^4(z+3)}$

7. $f(z) = cos\frac{z}{z^2(z-\pi)^3}$

8. $f(z) = \frac{e^z}{e^z - 1}$

9. $f(z) = sec\, z$

10. $f(z) = \dfrac{1}{z\, sin\, z}$

Use Cauchy's residue theorem, where appropriate, to evaluate the given integral along the indicated contours.

1. $1/((z-1)(z+2)^2)$ $(\boldsymbol{a})|z| = 1/2$ $(\boldsymbol{b})|z| = 3/2$ $(\boldsymbol{c})|z| = 3$

2. $\dfrac{z+1}{z^2(z-2i)}$ $(\boldsymbol{a})\ |z| = 1$ $(\boldsymbol{b})\ |z-2i| = 1$ $(\boldsymbol{c})\ |z-2i| = 4$

3. $\dfrac{z^3e^{-1}}{z^2}$ $(\boldsymbol{a})\ |z| = 5$ $(\boldsymbol{b})\ |z+i| = 2$ $(\boldsymbol{c})\ |z-3| = 1$

4. $\dfrac{1}{z sin\, z}$ $(\boldsymbol{a})\ |z-2i| = 1$ $(\boldsymbol{b})\ |z-2i| = 3$ $(\boldsymbol{c})\ |z| = 5$

5. $1/(z^2 + 4z + 13), C: |z-3i| = 3$

6. $\dfrac{z^3}{(z-1)^4}, C: |z-2| = 3/2$

7. $\dfrac{z}{z^4 - 1}, C: |z| = 2$

8. $\dfrac{z}{(z+1)(z^2+1)}$, C: $16x^2 + y^2 = 4$

9. $\dfrac{ze^z}{z^2 - 1}$, C: $|z| = 2$

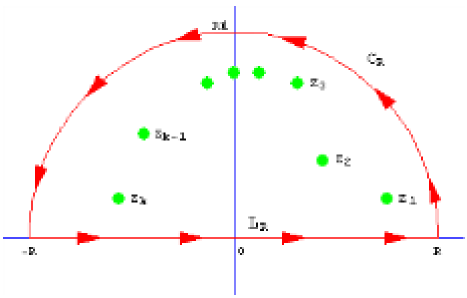
10. $\dfrac{e^z}{z^3 + 2z^2}$, C: $|z| = 3$

Lecture 24. Improper integrals of rational functions

Let $f(z) = \dfrac{P(z)}{Q(z)}$ where $P(x)$ and $Q(x)$ are polynomials, of degree m and n, respectively. If $Q(x) \neq 0$ for all real x and $n \geq m+2$, then the Cauchy Principal Value (or $P.V.$) of the integral is

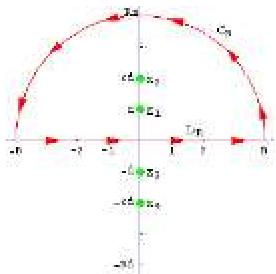
$$P.V.\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{j=1}^{k} \operatorname{Res}\left[\frac{P}{Q}, z_j\right].$$

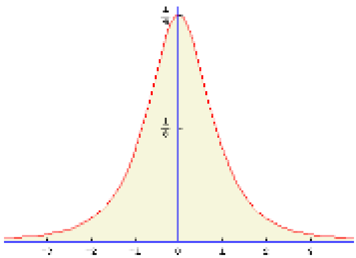
where $z_1, z_2, \ldots, z_k$ are the poles of $\dfrac{P(z)}{Q(z)}$ that lie in the upper half-plane,



Remark. The residues at the poles in the lower-half plane are not used in the computation.

Example. Use the residue calculus to compute $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\,dx$.





A portion of the area under the curve $f(x) = \frac{1}{(x^2+1)(x^2+4)}$, over the interval $(-\infty, \infty)$.

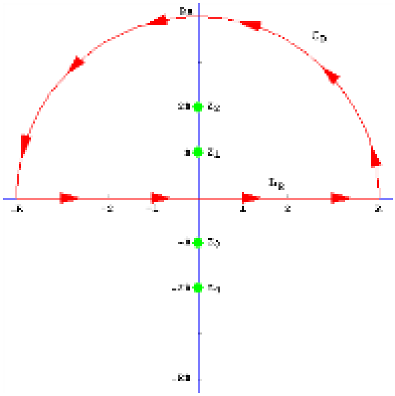
Solution.

We write the integrand as

$$f(z) = \frac{1}{(z^2+1)(z^2+4)} = \frac{1}{(z+i)(z-i)(z+2i)(z-2i)}$$

.

We see that $f(z)$ has simple poles at the points $z = \pm i$ and $z = \pm 2i$, and that the poles at $z_1 = i$ and $z_2 = 2i$, are the only singularities of $f(z)$ in the upper half-plane.



The contour $C = C_R + L_R$ consisting of the semi-circle C_R and the interval $L_R = \{x : -R \le x \le R\}$. The points $z_1 = i$, $z_2 = 2i$ lie in the upper half-plane. Computing the residues, we obtain

$$\begin{aligned}
\mathrm{Res}[f, z_1] &= \mathrm{Res}[f, i] \\
&= \lim_{z \to z_1} (z - z_1) f(z) \\
&= \lim_{z \to i} (z - i) \frac{1}{(z + i)(z - i)(z + 2i)(z - 2i)} \\
&= \lim_{z \to i} (z - i) \frac{1}{(z + i)(z - i)(z + 2i)(z - 2i)} \\
&= \lim_{z \to i} \frac{1}{(z + i)(z + 2i)(z - 2i)} \\
&= \frac{1}{(i + i)(i + 2i)(i - 2i)} \\
&= \frac{1}{(2i)(3i)(-i)} \\
&= -\frac{i}{6}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{Res}[f, z_2] &= \mathrm{Res}[f, 2i] \\
&= \lim_{z \to z_2} (z - z_2) f(z) \\
&= \lim_{z \to 2i} (z - 2i) \frac{1}{(z + i)(z - i)(z + 2i)(z - 2i)} \\
&= \lim_{z \to 2i} (z - 2i) \frac{1}{(z + i)(z - i)(z + 2i)(z - 2i)} \\
&= \lim_{z \to 2i} \frac{1}{(z + i)(z - i)(z + 2i)} \\
&= \frac{1}{(2i + i)(2i - i)(2i + 2i)} \\
&= \frac{1}{(3i)(i)(4i)} \\
&= \frac{i}{12}
\end{aligned}$$

we conclude that

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\,dx = \lim_{R\to\infty} \int_{C} \frac{1}{(z^2+1)(z^2+4)}\,dz$$

$$= \lim_{R\to\infty} \int_{C} \frac{1}{(z+i)(z-i)(z+2i)(z-2i)}\,dz$$

$$= 2\pi i \sum_{j=1}^{2} \mathrm{Res}\left[\frac{P}{Q}, z_j\right]$$

$$= 2\pi i\,(\mathrm{Res}[f, z_1] + \mathrm{Res}[f, z_2])$$

$$= 2\pi i\,(\mathrm{Res}[f, i] + \mathrm{Res}[f, 2i])$$

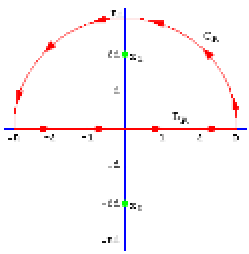
$$= 2\pi i\left(-\frac{i}{6} + \frac{i}{12}\right)$$

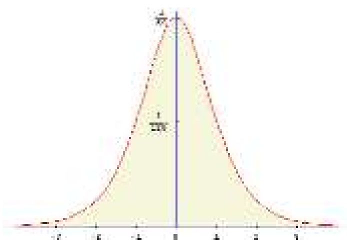
$$= 2\pi i\left(-\frac{i}{12}\right)$$

$$= \frac{1}{6}\pi$$

Example. Use the residue calculus to compute $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^3}\,dx$

.

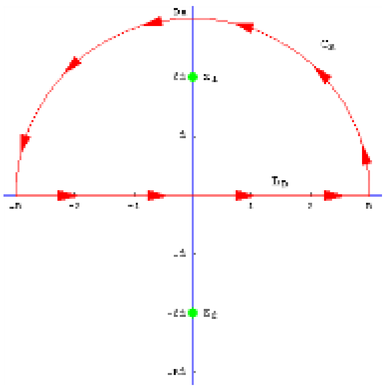




A portion of the area under the curve $f(x) = \dfrac{1}{(x^2+4)^3}$, over the interval $(-\infty, \infty)$.

Solution.

The integrand $f(z) = \frac{1}{(z^2+4)^3} = \frac{1}{(z+2i)^3(z-2i)^3}$ has a poles of order 3 at the points $z = \pm 2i$, and the poles a $z = 2i$ is the only singularity of $f(z)$ in the upper half-plane.



The contour $C = C_R + L_R$ consisting of the semi-circle C_R and the interval $L_R = \{x : -R \le x \le R\}$. The point $z_1 = 2i$ lies in the upper half-plane. Computing the residue at $z_1 = 2i$, we get

$$\begin{aligned}
\text{Res}[f, z_1] &= \text{Res}[f, 2\,i] \\
&= \frac{1}{2\,!} \lim_{z \to 2\,i} \frac{d^2}{dz^2} \left((z - 2\,i)^3 f(z) \right) \\
&= \frac{1}{2} \lim_{z \to 2\,i} \frac{d^2}{dz^2} \left((z - 2\,i)^3 \frac{1}{(z + 2\,i)^3 (z - 2\,i)^3} \right) \\
&= \frac{1}{2} \lim_{z \to 2\,i} \frac{d^2}{dz^2} \left(\frac{1}{(z + 2\,i)^3} \right) \\
&= \frac{1}{2} \lim_{z \to 2\,i} \frac{d}{dz} \left(\frac{-3}{(z + 2\,i)^4} \right) \\
&= \frac{1}{2} \lim_{z \to 2\,i} \left(\frac{12}{(z + 2\,i)^5} \right) \\
&= \frac{1}{2} \left(\frac{12}{(2\,i + 2\,i)^5} \right) \\
&= \frac{1}{2} \left(\frac{12}{(4\,i)^5} \right) \\
&= \frac{1}{2} \left(\frac{12}{1024\,i} \right) \\
&= -\frac{3\,i}{512}
\end{aligned}$$

we conclude that

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^3}\, dx = \lim_{R\to\infty} \int_C \frac{1}{(z^2+4)^3}\, dz$$

$$= \lim_{R\to\infty} \int_C \frac{1}{(z+2i)^3 (z-2i)^3}\, dz$$

$$= 2\pi i \sum_{j=1}^{1} \mathrm{Res}\left[\frac{P}{Q}, z_j\right]$$

$$= 2\pi i\, \mathrm{Res}[f, z_1]$$

$$= 2\pi i\, \mathrm{Res}[f, 2i]$$

$$= 2\pi i \left(-\frac{3i}{512}\right)$$

$$= \frac{3\pi}{256}$$

Lecture 25. Improper integrals involving trigonometric functions

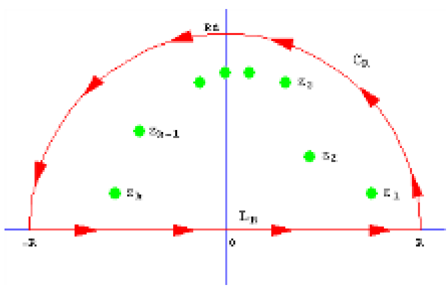
Assume that $P(x)$ and $Q(x)$ are polynomials with real coefficients, of degree m and n, respectively, where $n \geq m+1$ and $Q(x) \neq 0$ for all real x.

If $f(z) = \dfrac{P(z)}{Q(z)} e^{i\alpha z}$, where α is a real number satisfying $\alpha > 0$,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(\alpha x)\, dx = \operatorname{Re}\left(2\pi i \sum_{j=1}^{k} \operatorname{Res}[f, z_j]\right),$$

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(\alpha x)\, dx = \operatorname{Im}\left(2\pi i \sum_{j=1}^{k} \operatorname{Res}[f, z_j]\right),$$

where $z_1, z_2, \ldots, z_k$ are the poles of $f(z)$ that lie in the upper half-plane.

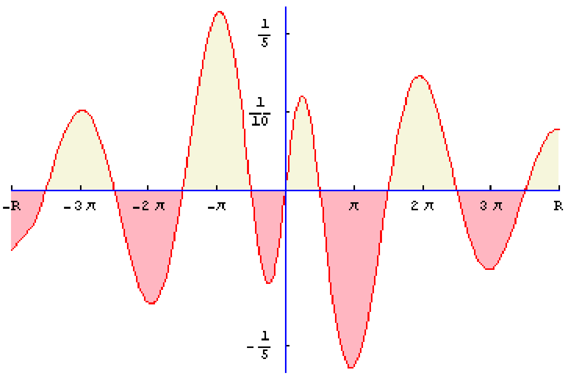


The poles $z_1, z_2, \ldots, z_k$ of $f(z) = \frac{P(z)}{Q(z)} e^{i\alpha z}$ in the upper half-plane.

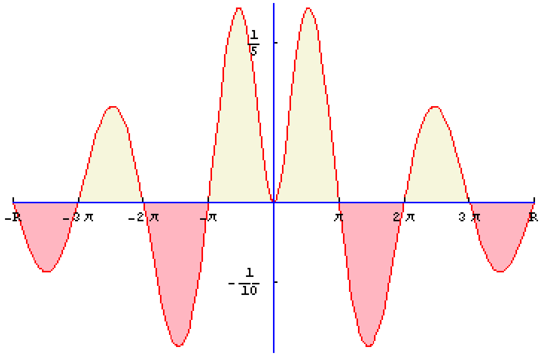
Remark 1. The residues at the poles in the lower-half plane are not used in the computation.

Example. Evaluate $\text{P.V.} \int_{-\infty}^{\infty} \frac{x\cos(x)}{x^2+4}\,dx$

and $\text{P.V.} \int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2+4}\,dx$.

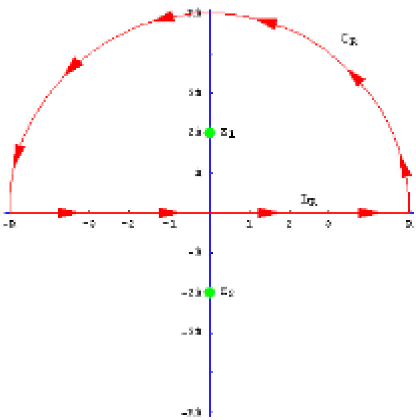


A portion of $f(x) = \frac{x\cos(x)}{x^2+4}$, over the interval $(-\infty, \infty)$.



A portion of $f(x) = \dfrac{x \sin(x)}{x^2 + 4}$, over the interval $(-\infty, \infty)$.

Solution. The function $f(z)$ in Equation (8-12) is $f(z) = \dfrac{z e^{iz}}{z^2 + 4} = \dfrac{z e^{iz}}{(z + 2i)(z - 2i)}$, which has a simple pole at the pole at $z_1 = 2i$ in the upper half-plane.



Here the denominator of $f(z)$ has a factor of the form $(z - z_1)$, and we can to calculate

$$\text{Res}[f, z_1] = \lim_{z \to z_1} (z - z_1) f(z)$$.

In this example, the limit can be calculated as follows:

$$\begin{aligned}
\text{Res}[f, z_1] &= \text{Res}[f, 2i] \\
&= \lim_{z \to z_1} (z - z_1) f(z) \\
&= \lim_{z \to 2i} (z - 2i) f(z) \\
&= \lim_{z \to 2i} (z - 2i) \frac{z e^{iz}}{(z + 2i)(z - 2i)} \\
&= \lim_{z \to 2i} \frac{z e^{iz}}{(z + 2i)} \\
&= \frac{2i e^{i2i}}{(2i + 2i)} \\
&= \frac{2i e^{-2}}{4i} \\
&= \frac{1}{2e^2}
\end{aligned}$$

For $\text{P.V.} \int_{-\infty}^{\infty} \frac{x \cos(x)}{x^2 + 4} \, dx$, use Equation (8-13) and get

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{x\cos(x)}{x^2+4}\,dx = \text{Re}\left(2\pi i\sum_{j=1}^{1}\text{Res}[f, z_j]\right)$$

$$= \text{Re}\,(2\pi i\,\text{Res}[f, z_1])$$

$$= \text{Re}\,(2\pi i\,\text{Res}[f, 2i])$$

$$= \text{Re}\left(2\pi i\,\frac{1}{2e^2}\right)$$

$$= \text{Re}\left(i\,\frac{\pi}{e^2}\right)$$

$$= 0$$

For $\text{P.V.}\int_{-\infty}^{\infty}\frac{x\sin(x)}{x^2+4}\,dx$, and get

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x \sin(x)}{x^2+4} dx = \text{Im}\left(2\pi i \sum_{j=1}^{1} \text{Res}[f, z_j]\right)$$

$$= \text{Im}\,(2\pi i\, \text{Res}[f, z_1])$$

$$= \text{Im}\,(2\pi i\, \text{Res}[f, 2 i])$$

$$= \text{Im}\left(2\pi i \frac{1}{2 e^2}\right)$$

$$= \text{Im}\left(i \frac{\pi}{e^2}\right)$$

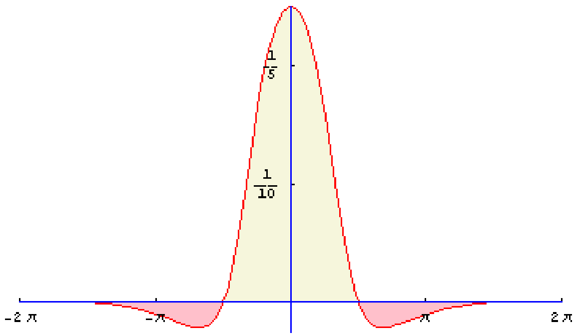
$$= \frac{\pi}{e^2}$$

Therefore,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x \cos(x)}{x^2+4} dx = 0$$

and $$\text{P.V.} \int_{-\infty}^{\infty} \frac{x \sin(x)}{x^2+4} dx = \frac{\pi}{e^2}.$$

Example. Evaluate $\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos(x)}{x^4+4}\,dx$

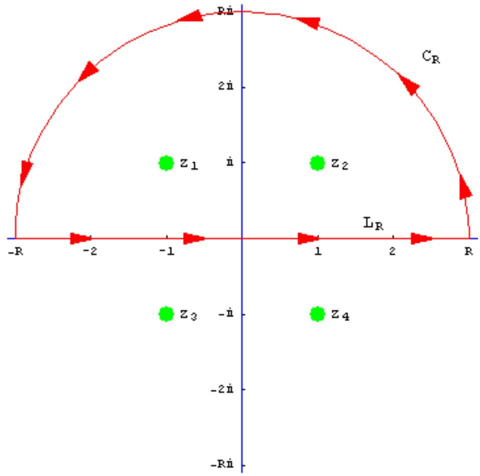


Solution. The function $f(z)$ in Equation (8-12) is

$$f(z) = \frac{e^{iz}}{z^4+4} = \frac{e^{iz}}{(z+1+i)(z+1-i)(z-1+i)(z-1-i)}$$

,

which has simple poles at the points $z_1 = -1+i$ and $z_2 = 1+i$ in the upper half-plane.



Here the denominator of $f(z)$ has a factor of the form $(z - z_k)$, and we can apply Theorem to calculate

$$\mathrm{Res}[f, z_k] = \lim_{z \to z_k} (z - z_k) f(z)$$

, for $k = 1, 2$.

In this example, the limits can be calculated with the assistance of L'Hôpital's rule:

$$\text{Res}[f, -1+i] = \lim_{z \to -1+i} (z+1-i)\, f(z)$$

$$= \lim_{z \to -1+i} \frac{(z+1-i)\, e^{iz}}{(z+1+i)(z+1-i)(z-1+i)(z-1-i)}$$

$$= \lim_{z \to -1+i} \frac{(z+1-i)\, e^{iz}}{z^4+4} \quad = \quad '\frac{0}{0}'$$

$$= \lim_{z \to -1+i} \frac{(1)\, e^{iz} + (z+1-i)\, i\, e^{iz}}{4z^3}$$

$$= \lim_{z \to -1+i} \frac{(1+i(z+1-i))\, e^{iz}}{4z^3}$$

$$= \frac{(1+i(-1+i+1-i))\, e^{i(-1+i)}}{4(-1+i)^3}$$

$$= \frac{(1)\, e^{-1-i}}{4(-1+i)^3}$$

$$= \frac{e^{-1-i}}{8+8i}$$

$$= \frac{(1-i)\, e^{-1-i}}{16}$$

Similarly,

$$\text{Res}[f, 1+i] = \lim_{z\to 1+i} (z-1-i)\, f(z)$$

$$= \lim_{z\to 1+i} \frac{(z-1-i)\, e^{iz}}{(z+1+i)(z+1-i)(z-1+i)(z-1-i)}$$

$$= \lim_{z\to 1+i} \frac{(z-1-i)\, e^{iz}}{z^4+4} = \text{'}\,\frac{0}{0}\,\text{'}$$

$$= \lim_{z\to 1+i} \frac{(1)\, e^{iz} + (z-1-i)\, i\, e^{iz}}{4z^3}$$

$$= \lim_{z\to 1+i} \frac{(1+i(z-1-i))\, e^{iz}}{4z^3}$$

$$= \frac{(1+i(1+i-1-i))\, e^{i(1+i)}}{4(1+i)^3}$$

$$= \frac{(1)\, e^{-1+i}}{4(1+i)^3}$$

$$= \frac{e^{-1+i}}{-8+8i}$$

$$= \frac{(-1-i)\, e^{-1+i}}{16}$$

Using Equation, we get

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos(x)}{x^4+4}\,dx = \text{Re}\left(2\pi i \sum_{j=1}^{2} \text{Res}[f, z_j]\right)$$

$$= \text{Re}\,(2\pi i\,(\text{Res}[f, z_1] + \text{Res}[f, z_2]))$$

$$= \text{Re}\,(2\pi i\,(\text{Res}[f, -1+i] + \text{Res}[f, 1+i]))$$

$$= \text{Re}\left(2\pi i\left(\left(\frac{1}{16} - \frac{i}{16}\right)e^{-1-i} + \left(-\frac{1}{16} - \frac{i}{16}\right)e^{-1+i}\right)\right)$$

$$= \text{Re}\left(2\pi i\left(-\frac{i}{8e}\left(\frac{e^{i}+e^{-i}}{2} + \frac{e^{i}-e^{-i}}{2i}\right)\right)\right)$$

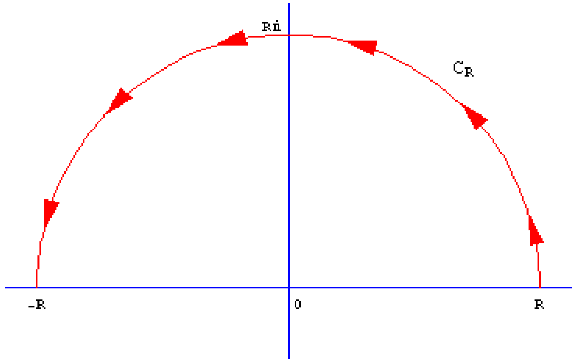
$$= \text{Re}\left(2\pi i\left(-\frac{i}{8e}\,(\cos(1) + \sin(1))\right)\right)$$

$$= \text{Re}\left(\frac{2\pi}{8e}\,(\cos(1) + \sin(1))\right)$$

$$= \frac{\pi\,(\cos(1) + \sin(1))}{4e}$$

Lemma (Jordan's Lemma). Assume that $P(x)$ and $Q(x)$ are polynomials with real coefficients, of degree m and n, respectively, where $n \geq m+1$ and that α is a real number satisfying $\alpha > 0$. If C_R is the upper semi-circle $z = Re^{i\theta}$ for $0 \leq \theta \leq \pi$, then

$$\lim_{R\to\infty} \int_{C_R} \frac{P(z)}{Q(z)}\,e^{i\alpha z}\,dz = 0 .$$



The upper semi circle $C_R = \{z = R e^{i\theta} : 0 \le \theta \le \pi\}$ in Jordan's

Theorem Contour Integration for Improper Trigonometric Integrals). Assume that $P(x)$ and $Q(x)$ are polynomials with real coefficients, of degree m and n, respectively, where $n \ge m+1$ and $Q(x) \ne 0$ for all real x.

If $f(z) = \frac{P(z)}{Q(z)} e^{i\alpha z}$, where α is a real number satisfying $\alpha > 0$, then

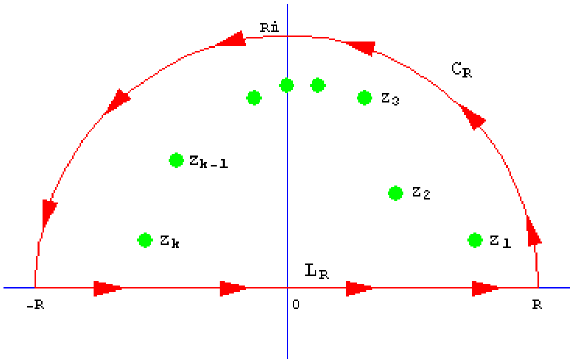
$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(\alpha x)\, dx = \text{Re}\left(2\pi i \sum_{j=1}^{k} \text{Res}[f, z_j]\right) = -2\pi \sum_{j=1}^{k} \text{Im}(\text{Res}[f, z_j])$$

,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(\alpha x)\, dx = \text{Im}\left(2\pi i \sum_{j=1}^{k} \text{Res}[f, z_j]\right) = 2\pi \sum_{j=1}^{k} \text{Re}(\text{Res}[f, z_j])$$

,

where $z_1, z_2, \ldots, z_k$ are the poles of $f(z)$ that lie in the upper half-plane, and $\mathrm{Re}(\mathrm{Res}[f, z_j])$ and $\mathrm{Im}(\mathrm{Res}[f, z_j])$ are the real and imaginary parts of $\mathrm{Res}[f, z_j]$.



The poles $z_1, z_2, \ldots, z_k$ of $\dfrac{P(z)}{Q(z)} e^{i\alpha z}$ that lie in the upper half-plane.

Remark. The residues at the poles in the lower-half plane are not used in the computation.

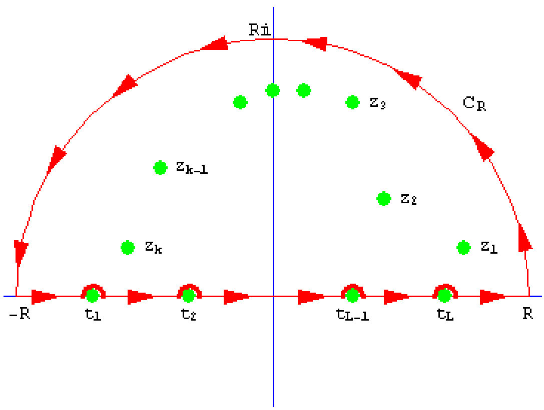
Lecture 26. Intended contour integrals

Assume that $f(z) = \dfrac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomials with real coefficients of degree m and n, respectively, and $n \geq m+2$. If $Q(z)$ has simple zeros at the points $t_1, t_2, \ldots, t_L$ on the x-axis, then

$$\mathrm{P.V.}\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{j=1}^{k} \mathrm{Res}\left[\frac{P}{Q}, z_j\right] + \pi i \sum_{j=1}^{L} \mathrm{Res}\left[\frac{P}{Q}, t_j\right]$$

,

where $z_1, z_2, \ldots, z_k$ are the poles of $f(z)$ that lie in the upper half-plane.



The poles $t_1, t_2, \ldots, t_L$ of $f(z)$ that lie on the x-axis, and the poles $z_1, z_2, \ldots, z_k$ that lie in the upper half-plane above the semicircles $C_1, C_2, \ldots, C_L$.

Theorem (Indented Trigonometric Integrals). Assume that $P(z)$ and $Q(z)$ are polynomials with real coefficients of degree m and n, respectively, where $n \geq m+1$ and that $Q(z)$ has simple zeros at the points $t_1, t_2, \ldots, t_L$ on the x-axis. If α is a positive real number, and if $f(z) = \dfrac{e^{i\alpha z} P(z)}{Q(z)}$, then we can compute the Cauchy Principal Value (P.V.) of the following integrals

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(\alpha x)\, dx = \text{Re}\left(2\pi i \sum_{j=1}^{k} \text{Res}[f, z_j]\right) + \text{Re}\left(\pi i \sum_{j=1}^{L} \text{Res}[f, t_j]\right),$$

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(\alpha x)\, dx = \text{Im}\left(2\pi i \sum_{j=1}^{k} \text{Res}[f, z_j]\right) + \text{Im}\left(\pi i \sum_{j=1}^{L} \text{Res}[f, t_j]\right),$$

where $z_1, z_2, \ldots, z_k$ are the poles of $f(z)$ that lie in the upper half plane.

Remark 1. The residues at the poles in the lower half-plane are not used in the computation.

Remark 2. If you prefer, you can use the alternative way to write formulas

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\cos(\alpha x)\,dx = -2\pi\sum_{j=1}^{k}\text{Im}(\text{Res}[f, z_j]) - \pi\sum_{j=1}^{L}\text{Im}(\text{Res}[f, t_j])$$

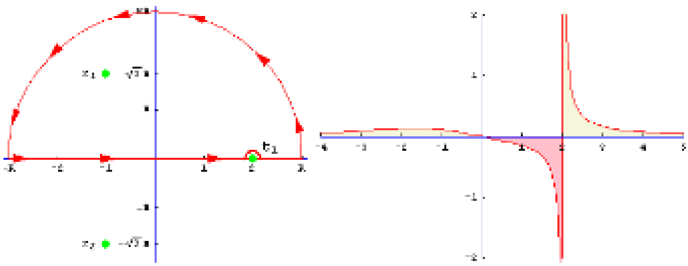
,

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\sin(\alpha x)\,dx = 2\pi\sum_{j=1}^{k}\text{Re}(\text{Res}[f, z_j]) + \pi\sum_{j=1}^{L}\text{Re}(\text{Res}[f, t_j])$$

.

Remark 3. The formulas in these theorems use the Cauchy principal value of the integral, which pays special attention to the manner in which any limits are taken. we add one-half of the value of each residue at the points $t_1, t_2, \ldots, t_r$ on the x-axis.

Example 8.20. Evaluate $\text{P.V.}\int_{-\infty}^{\infty}\frac{x}{x^3-8}\,dx$ by using the residue calculus.



Solution. The integrand

$$f(z) = \frac{P(z)}{Q(z)}$$

$$= \frac{z}{z^3 - 8}$$

$$= \frac{z}{(z-2)\,(z^2 + 2z + 4)}$$

$$= \frac{z}{(z-2)\left(z + 1 + i\sqrt{3}\right)\left(z + 1 - i\sqrt{3}\right)}$$

has simple poles at the point $t_1 = 2$ on the x-axis, and at $z_1 = -1 + i\sqrt{3}$ in the upper half-plane. Here the denominator of $f(z)$ has factors of the form $(z - t_1)$, and $(z - z_1)$, and we can

$$\mathrm{Res}[f, t_1] = \lim_{z \to t_1} (z - t_1)\, f(z),$$

and

$$\mathrm{Res}[f, z_1] = \lim_{z \to z_1} (z - z_1)\, f(z).$$

In this example, the limits can be calculated as follows:

$$\text{Res}[f, 2] = \lim_{z \to 2} (z-2)\, f(z)$$

$$= \lim_{z \to 2} (z-2) \frac{z}{(z-2)(z^2 + 2z + 4)}$$

$$= \lim_{z \to 2} \frac{z}{(z^2 + 2z + 4)}$$

$$= \frac{2}{(2^2 + 2 \times 2 + 4)}$$

$$= \frac{2}{(4 + 4 + 4)}$$

$$= \frac{1}{6}$$

$$\mathrm{Res}\left[f, -1 + i\sqrt{3}\right] = \lim_{z\to -1+i\sqrt{3}} \left(z + 1 - i\sqrt{3}\right) f(z)$$

$$= \lim_{z\to -1+i\sqrt{3}} \left(z + 1 - i\sqrt{3}\right) \frac{z}{(z-2)\left(z+1+i\sqrt{3}\right)\left(z+1-i\sqrt{3}\right)}$$

$$= \lim_{z\to -1+i\sqrt{3}} \frac{z}{(z-2)\left(z+1+i\sqrt{3}\right)}$$

$$= \frac{-1+i\sqrt{3}}{\left(-1+i\sqrt{3}-2\right)\left(-1+i\sqrt{3}+1+i\sqrt{3}\right)}$$

$$= \frac{-1+i\sqrt{3}}{\left(-3+i\sqrt{3}\right)\left(2i\sqrt{3}\right)}$$

$$= \frac{-1+i\sqrt{3}}{-6-6i\sqrt{3}}$$

$$= \frac{\left(-1+i\sqrt{3}\right)\left(-6+6i\sqrt{3}\right)}{\left(-6-6i\sqrt{3}\right)\left(-6+6i\sqrt{3}\right)}$$

$$= \frac{-12-12i\sqrt{3}}{144}$$

$$= \frac{-1-i\sqrt{3}}{12}$$

Using Theorem (Indented Integrals), $\mathrm{P.V.} \int_{-\infty}^{\infty} \frac{x}{x^3 - 8}\, dx$ is computed with

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x}{x^3 - 8}\,dx = 2\pi i \,\text{Res}\left[\frac{P}{Q}, z_1\right] + \pi i \,\text{Res}\left[\frac{P}{Q}, t_1\right]$$

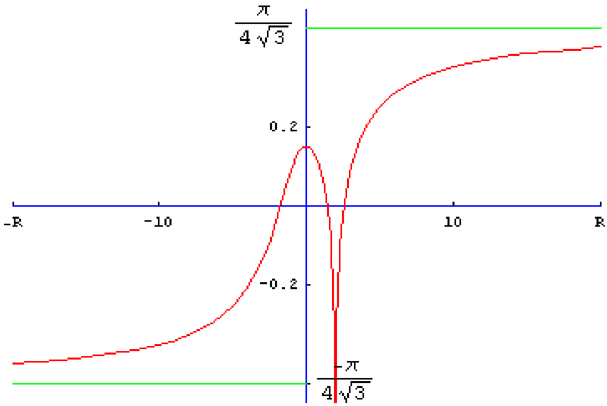
$$= 2\pi i \left(\frac{-1 - i\sqrt{3}}{12}\right) + \pi i \left(\frac{1}{6}\right)$$

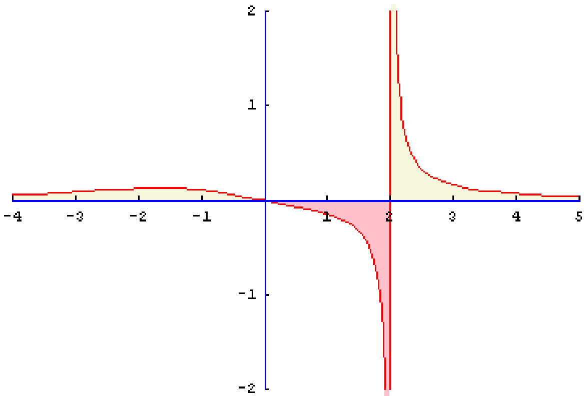
$$= \pi i \left(\frac{-1 - i\sqrt{3}}{6}\right) + \pi i \left(\frac{1}{6}\right)$$

$$= \pi i \left(\frac{- i\sqrt{3}}{6}\right)$$

$$= \frac{\sqrt{3}}{6}\pi$$

Example. Evaluate $\text{P.V.} \int_{-\infty}^{\infty} \frac{t}{t^3 - 8}\,dt$ by using a computer algebra system.





Solution. Computer algebra systems such as *Mathematica*™ or Maple™ give the indefinite integral

$$\int \frac{t}{t^3-8}\,dt = \frac{\operatorname{Arctan}\left(\frac{1+t}{\sqrt{3}}\right)}{2\sqrt{3}} + \frac{\operatorname{Log}(t-2)}{6} - \frac{\operatorname{Log}(t^2+2t+4)}{12} = g(t)$$

.

For real numbers, we must write the second term as $\frac{\mathrm{Log}[(t-2)^2]}{12}$ and use the equivalent formula:

$$g(t) = \frac{\mathrm{Arctan}\left(\frac{1+t}{\sqrt{3}}\right)}{2\sqrt{3}} + \frac{\mathrm{Log}[(t-2)^2]}{12} - \frac{\mathrm{Log}(t^2+2t+4)}{12}$$

$$= \frac{\mathrm{Arctan}\left(\frac{1+t}{\sqrt{3}}\right)}{2\sqrt{3}} + \frac{\mathrm{Log}[(t-2)^2]}{12} - \frac{\mathrm{Log}[(t+2)^2]}{12}$$

$$= \frac{\mathrm{Arctan}\left(\frac{1+t}{\sqrt{3}}\right)}{2\sqrt{3}} + \frac{1}{12}\,\mathrm{Log}\left[\frac{(t-2)^2}{(t+2)^2}\right]$$

$$= \frac{\mathrm{Arctan}\left(\frac{1+t}{\sqrt{3}}\right)}{2\sqrt{3}} + \frac{1}{12}\,\mathrm{Log}\left[\left(\frac{t-2}{t+2}\right)^2\right]$$

This antiderivative has the property that $\lim_{t\to 2} g(t) = -\infty$, as shown in Figure. We also compute

$$\lim_{t\to\infty} g(t) = \lim_{t\to\infty}\left(\frac{\mathrm{Arctan}\left(\frac{1+t}{\sqrt{3}}\right)}{2\sqrt{3}} + \frac{1}{12}\,\mathrm{Log}\left[\left(\frac{t-2}{t+2}\right)^2\right]\right)$$

$$= \lim_{t\to\infty}\frac{\mathrm{Arctan}\left(\frac{1+t}{\sqrt{3}}\right)}{2\sqrt{3}} + \lim_{t\to\infty}\frac{1}{12}\,\mathrm{Log}\left[\left(1-\frac{4}{t+2}\right)^2\right]$$

$$= \frac{\frac{\pi}{2}}{2\sqrt{3}} + \frac{1}{12}\,\mathrm{Log}[(1-0)^2]$$

$$= \frac{\pi\sqrt{3}}{12}$$

$$\lim_{t\to-\infty} g(t) = \lim_{t\to-\infty}\left(\frac{\mathrm{Arctan}\left(\frac{1+t}{\sqrt{3}}\right)}{2\sqrt{3}} + \frac{1}{12}\,\mathrm{Log}\left[\left(\frac{t-2}{t+2}\right)^2\right]\right)$$

$$= \lim_{t\to-\infty}\frac{\mathrm{Arctan}\left(\frac{1+t}{\sqrt{3}}\right)}{2\sqrt{3}} + \lim_{t\to-\infty}\frac{1}{12}\,\mathrm{Log}\left[\left(1-\frac{4}{t+2}\right)^2\right]$$

$$= \frac{-\frac{\pi}{2}}{2\sqrt{3}} + \frac{1}{12}\,\mathrm{Log}[(1-0)^2]$$

$$= -\frac{\pi\sqrt{3}}{12}$$

The Cauchy principal limit at $t = 2$ as $r \to 0$ is

$$\lim_{r\to 0+}(g(2+r)-g(2-r)) = \lim_{r\to 0+}\left(\frac{\text{ArcTan}\frac{3+r}{\sqrt{3}}}{2\sqrt{3}} + \frac{\text{Log}\frac{r^2}{(4+r)^2}}{12} - \frac{\text{ArcTan}\frac{3-r}{\sqrt{3}}}{2\sqrt{3}} - \frac{\text{Log}\frac{r^2}{(4-r)^2}}{12}\right)$$

$$= \lim_{r\to 0+}\left(\frac{\text{ArcTan}\frac{3+r}{\sqrt{3}}}{2\sqrt{3}} - \frac{\text{ArcTan}\frac{3-r}{\sqrt{3}}}{2\sqrt{3}} + \frac{1}{12}\text{Log}\frac{(4-r)^2}{(4+r)^2}\right)$$

$$= \frac{\text{ArcTan}\frac{3}{\sqrt{3}}}{2\sqrt{3}} - \frac{\text{ArcTan}\frac{3}{\sqrt{3}}}{2\sqrt{3}} + \frac{1}{12}\text{Log}(1)$$

$$= 0$$

Therefore the Cauchy principal value of the improper integral is

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{t}{t^3-8}\,dt = \lim_{r\to 0+}\left(\int_{-\infty}^{2-r}\frac{t}{t^3-8}\,dt + \int_{2+r}^{\infty}\frac{t}{t^3-8}\,dt\right)$$

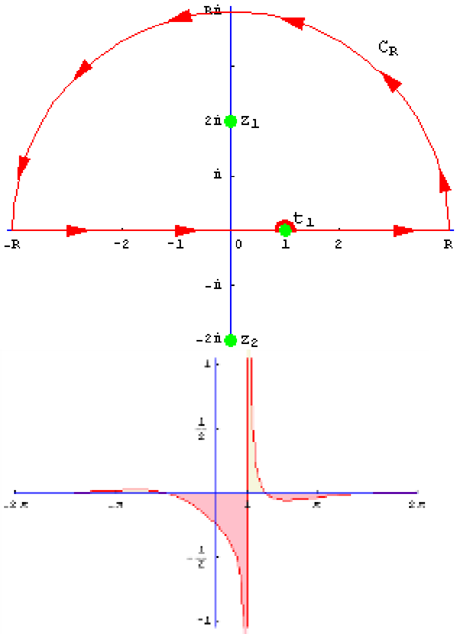
$$= \lim_{t\to\infty} g(t) - \lim_{r\to 0+}(g(2+r)-g(2-r)) - \lim_{t\to -\infty} g(t)$$

$$= \frac{\pi\sqrt{3}}{12} - 0 - \frac{-\pi\sqrt{3}}{12}$$

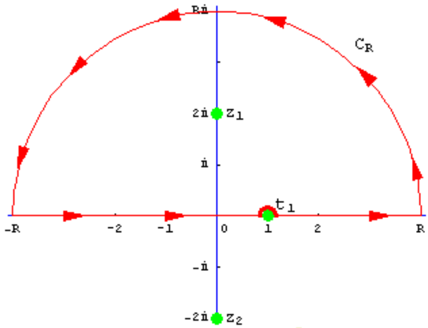
$$= \frac{\pi\sqrt{3}}{6}$$

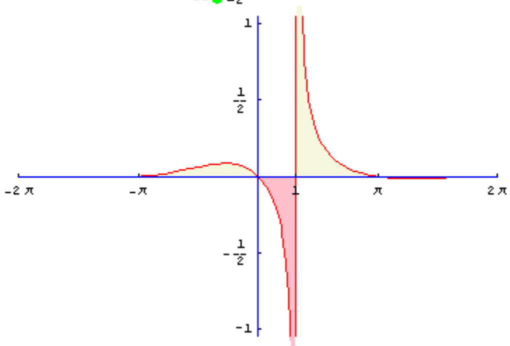
$$\text{P.V.}\int_{-\infty}^{\infty}\frac{\cos(x)}{(x-1)(x^2+4)}\,dx = \frac{1}{10}\pi\left(-\frac{1}{e^2} - 2\sin(1)\right)$$

.



$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin(x)}{(x-1)(x^2+4)}\, dx = \frac{1}{5}\pi\left(-\frac{1}{e^2} + \cos(1)\right).$$





Solution. The integrand

$$f(z) = \frac{e^{iz}}{(z-1)(z^2+4)} = \frac{e^{iz}}{(z-1)(z+2i)(z-2i)}$$ has

simple poles at the point $t_1 = 1$ on the x-axis, and at the point $z_1 = 2i$ in the upper half-plane. Here the denominator of $f(z)$ has factors of the form $(z - t_1)$, and $(z - z_1)$, and we can calculate

$$\mathrm{Res}[f, t_1] = \lim_{z \to t_1} (z - t_1) f(z)$$, and

$$\mathrm{Res}[f, z_1] = \lim_{z \to z_1} (z - z_1) f(z)$$.

In this example, the limits can be calculated as follows:

$$\mathrm{Res}[f, 1] = \lim_{z \to 1} (z - 1) f(z)$$

$$= \lim_{z \to 1} (z - 1) \frac{e^{i z}}{(z - 1) (z^2 + 4)}$$

$$= \lim_{z \to 1} \frac{e^{i z}}{(z^2 + 4)}$$

$$= \frac{e^{i}}{(1^2 + 4)}$$

$$= \frac{\cos(1) + i \sin(1)}{5}$$

and

$$\text{Res}[f, 2i] = \lim_{z\to 2i} (z - 2i)\, f(z)$$

$$= \lim_{z\to 2i} (z - 2i) \frac{e^{iz}}{(z-1)(z+2i)(z-2i)}$$

$$= \lim_{z\to 2i} \frac{e^{iz}}{(z-1)(z+2i)}$$

$$= \frac{e^{i(2i)}}{(2i-1)(2i+2i)}$$

$$= \frac{e^{-2}}{(-8-4i)}$$

$$= \frac{e^{-2}(-8+4i)}{(-8-4i)(-8+4i)}$$

$$= \frac{-8+4i}{80e^2}$$

$$= \frac{-2+i}{20e^2}$$

Using Theorem (<u>Indented Trig. Integrals</u>), $\text{P.V.}\int_{-\infty}^{\infty} \frac{\cos(x)}{(x-1)(x^2+4)}\,dx$ is computed

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{\cos(x)}{(x-1)\,(x^2+4)}\,dx = \text{Re}\left(2\pi i\sum_{j=1}^{1}\text{Res}[f, z_j]\right) + \text{Re}\left(\pi i\sum_{j=1}^{1}\text{Res}[f, t_j]\right)$$

$$= \text{Re}\,(2\pi i\,\text{Res}[f, z_1]) + \text{Re}\,(\pi i\,\text{Res}[f, t_1])$$

$$= \text{Re}\,(2\pi i\,\text{Res}[f, 2i]) + \text{Re}\,(\pi i\,\text{Res}[f, 1])$$

$$= \text{Re}\left(2\pi i\,\frac{-2+i}{20e^2}\right) + \text{Re}\left(\pi i\,\frac{\cos(1) + i\sin(1)}{5}\right)$$

$$= \text{Re}\left(2\pi i\left(-\frac{1}{10e^2} + i\,\frac{1}{20e^2}\right)\right) + \text{Re}\left(\pi i\left(\frac{1}{5}\cos(1) + \frac{1}{5}\,i\sin(1)\right)\right)$$

$$= 2\pi\,\text{Re}\left(-\frac{1}{20e^2} - i\,\frac{1}{10e^2}\right) + \pi\,\text{Re}\left(-\frac{1}{5}\sin(1) + \frac{1}{5}\,i\cos(1)\right)$$

$$= 2\pi\left(-\frac{1}{20e^2}\right) + \pi\left(-\frac{1}{5}\sin(1)\right)$$

$$= -\frac{\pi}{10e^2} - \frac{\pi\sin(1)}{5}$$

$$= \frac{1}{10}\pi\left(-\frac{1}{e^2} - 2\sin(1)\right)$$

Using Theorem (<u>Indented Trig. Integrals</u>), $\text{P.V.}\int_{-\infty}^{\infty} \frac{\sin(x)}{(x-1)\,(x^2+4)}\,dx$ is computed.

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin(x)}{(x-1)(x^2+4)}\, dx = \operatorname{Im}\left(2\pi i \sum_{j=1}^{1} \operatorname{Res}[f, z_j]\right) + \operatorname{Im}\left(\pi i \sum_{j=1}^{1} \operatorname{Res}[f, t_j]\right)$$

$$= \operatorname{Im}(2\pi i \operatorname{Res}[f, z_1]) + \operatorname{Im}(\pi i \operatorname{Res}[f, t_1])$$

$$= \operatorname{Im}(2\pi i \operatorname{Res}[f, 2i]) + \operatorname{Im}(\pi i \operatorname{Res}[f, 1])$$

$$= \operatorname{Im}\left(2\pi i \frac{-2+i}{20 e^2}\right) + \operatorname{Im}\left(\pi i \frac{\cos(1) + i \sin(1)}{5}\right)$$

$$= \operatorname{Im}\left(2\pi i \left(-\frac{1}{10 e^2} + i \frac{1}{20 e^2}\right)\right) + \operatorname{Im}\left(\pi i \left(\frac{1}{5}\cos(1) + \frac{1}{5} i \sin(1)\right)\right)$$

$$= 2\pi \operatorname{Im}\left(-\frac{1}{20 e^2} - i \frac{1}{10 e^2}\right) + \pi \operatorname{Im}\left(-\frac{1}{5}\sin(1) + \frac{1}{5} i \cos(1)\right)$$

$$= 2\pi\left(\frac{1}{10 e^2}\right) + \pi\left(\frac{1}{5}\cos(1)\right)$$

$$= -\frac{\pi}{5 e^2} + \frac{\pi \cos(1)}{5}$$

$$= \frac{1}{5}\pi\left(-\frac{1}{e^2} + \cos(1)\right)$$

Final Remarks.

Lemma. Assume that $f(z)$ has a simple pole at the point t_0 on the x-axis. If the contour is the upper semi-circle $C_r = \{z = t_0 + r e^{i\theta} : 0 \le \theta \le \pi\}$, then

$$\lim_{r \to 0} \int_{C_r} f(z)\, dz = i \pi \operatorname{Res}[f, t_0]$$

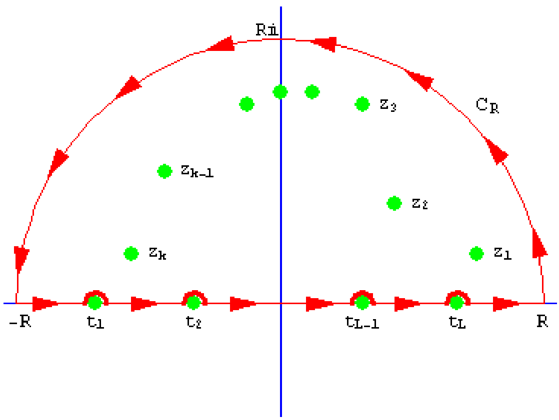
.

Theorem (Indented Integrals). Assume that $f(z) = \dfrac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomials with real coefficients of degree m and n, respectively, and $n \geq m+2$. If $Q(z)$ has simple zeros at the points $t_1, t_2, \ldots, t_L$ on the x-axis, then

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum_{j=1}^{k} \text{Res}\left[\frac{P}{Q}, z_j\right] + \pi i \sum_{j=1}^{L} \text{Res}\left[\frac{P}{Q}, t_j\right]$$

,

where $z_1, z_2, \ldots, z_k$ are the poles of $f(z)$ that lie in the upper half-plane.



The poles $t_1, t_2, \ldots, t_L$ of $f(z)$ that lie on the x-axis, and the poles $z_1, z_2, \ldots, z_k$ that lie in the upper half-plane above the semicircles $C_1, C_2, \ldots, C_L$.

Theorem (Indented Trigonometric Integrals). Assume that $P(z)$ and $Q(z)$ are polynomials with real coefficients of degree m and n, respectively, where $n \geq m+1$ and that $Q(z)$ has simple zeros at the points $t_1, t_2, \ldots, t_L$ on the x-axis. If α is a positive real number, and if $f(z) = \frac{e^{i\alpha z} P(z)}{Q(z)}$, then we can compute the Cauchy Principal Value (P.V.) of the following integrals

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(\alpha x)\, dx = \text{Re}\left(2\pi i \sum_{j=1}^{k} \text{Res}[f, z_j]\right) + \text{Re}\left(\pi i \sum_{j=1}^{L} \text{Res}[f, t_j]\right)$$

,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(\alpha x)\, dx = \text{Im}\left(2\pi i \sum_{j=1}^{k} \text{Res}[f, z_j]\right) + \text{Im}\left(\pi i \sum_{j=1}^{L} \text{Res}[f, t_j]\right)$$

,

where $z_1, z_2, \ldots, z_k$ are the poles of $f(z)$ that lie in the upper half plane.

Remark 1. The residues at the poles in the lower half-plane are not used in the computation.

Remark 2. If you prefer, you can use the alternative way to write formulas

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(\alpha x)\, dx = -2\pi \sum_{j=1}^{k} \text{Im}(\text{Res}[f, z_j]) - \pi \sum_{j=1}^{L} \text{Im}(\text{Res}[f, t_j])$$

,

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\sin(\alpha x)\,dx = 2\pi\sum_{j=1}^{k}\mathrm{Re}\,(\mathrm{Res}[f, z_j]) + \pi\sum_{j=1}^{L}\mathrm{Re}\,(\mathrm{Res}[f, t_j])$$

Remark 3. The formulas in these theorems give the Cauchy principal value of the integral, which pays special attention to the manner in which any limits are taken. .

Lecture 27. Integrands with Branch Points

We now show how to evaluate certain improper real integrals involving the integrand $\frac{x^{\alpha}P(x)}{Q(x)}$. Since the complex function z^{α} is multivalued, so we must first specify the branch to be used.

Let α be a real number with $0 < \alpha < 1$. In this section we use the branch of z^{α} corresponding to the branch of the logarithm $\log_0(z)$ as follows:

$$z^{\alpha} = e^{\alpha(\log_0(z))} = e^{\alpha(\ln|z| + i\arg_0 z)} = e^{\alpha(\ln r + i\theta)} = r^{\alpha}(\cos\alpha\theta + i\sin\alpha\theta)$$

,

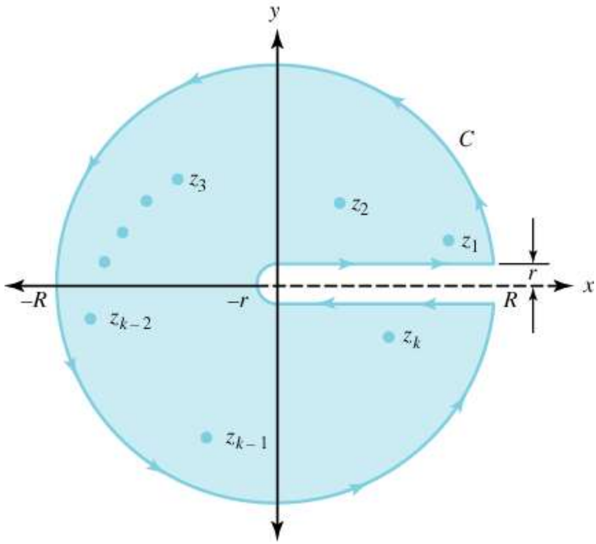
where $z = re^{i\theta}$ and $0 < \theta \le 2\pi$. Note that this is not the traditional principal branch of z^{α} and that, as defined, the function z^{α} is analytic in the domain $\{z = re^{i\theta} : r > 0,\ 0 < \theta < 2\pi\}$.

Theorem. Let P(x) and Q(x) be polynomials of degree m and n, respectively, where $n \ge m + 2$. If $Q(x) \ne 0$ for $x > 0$, and Q(x) has a zero of order at most 1 at the origin, and

$f(z) = \dfrac{z^{\alpha} P(z)}{Q(z)}$, where $0 < \alpha < 1$, then

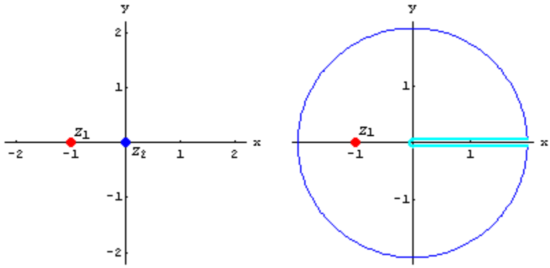
$$\text{P.V.} \int_0^{\infty} \frac{x^{\alpha} P(x)}{Q(x)}\, dx = \frac{2\pi i}{1 - e^{i\alpha 2\pi}} \sum_{j=1}^{k} \operatorname{Res}[f, z_j],$$

where $z_1, z_2, \ldots, z_k$ are the non-zero poles of $\dfrac{P(z)}{Q(z)}$.



The contour C that encloses the nonzero poles $z_1, z_2, \ldots, z_k$ of $\dfrac{P(z)}{Q(z)}$.

Example. Evaluate $\text{P.V.} \displaystyle\int_0^{\infty} \frac{x^{\alpha}}{x(x+1)}\, dx$, where $0 < \alpha < 1$.



Solution. The function $f(z) = \dfrac{x^{\alpha}}{x\,(x+1)}$ has a nonzero pole at the point $z_1 = -1$, and the denominator has a zero of order at most 1 (in fact, exactly 1) at the origin. Calculating the residue we get

$$\text{Res}[f, -1] = \lim_{z \to -1} (z+1)\, f(z)$$

$$= \lim_{z \to -1} (z+1) \frac{x^{\alpha}}{x\,(x+1)}$$

$$= \lim_{z \to -1} \frac{x^{\alpha}}{x} = \frac{(-1)^{\alpha}}{-1}$$

$$= \frac{(-1)^{\alpha}}{-1} = \frac{\left(e^{i\pi}\right)^{\alpha}}{-1}$$

$$= \frac{e^{i\alpha\pi}}{-1}$$

Using Theorem 8.7, we have

$$\text{P.V.}\int_0^\infty \frac{x^\alpha}{x\,(x+1)}\,dx = \frac{2\pi i}{1-e^{i\alpha 2\pi}}\,\text{Res}[f, z_1]$$

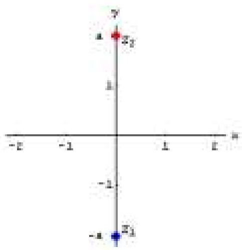
$$= \frac{2\pi i}{1-e^{i\alpha 2\pi}}\,\text{Res}[f, -1]$$

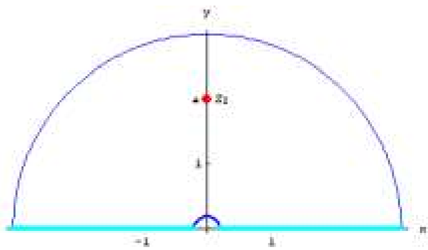
$$= \frac{2\pi i}{1-e^{i\alpha 2\pi}}\left(\frac{e^{i\alpha\pi}}{-1}\right) = \frac{2\pi i}{e^{i\alpha\pi}-e^{-i\alpha\pi}}$$

$$= \frac{\pi}{\frac{e^{i\alpha\pi}-e^{-i\alpha\pi}}{2i}}$$

$$= \frac{\pi}{\sin\alpha\pi}$$

Example. Evaluate $\text{P.V.}\int_0^\infty \frac{\ln x}{x^2+a^2}\,dx = \frac{\pi\,\text{Log}[a]}{2a}$, where $a > 0$.



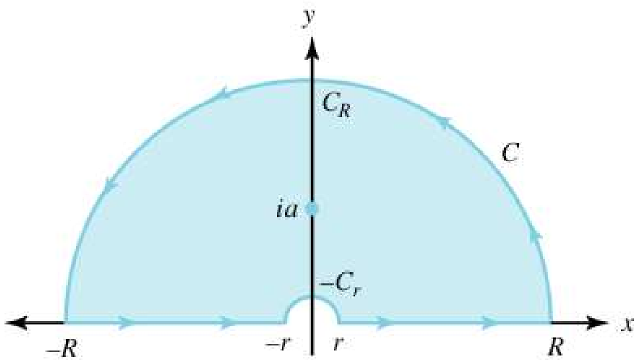


Solution. We use the function $f(z) = \dfrac{\log_{-\frac{\pi}{2}}(z)}{z^2 + a^2}$. Recall that

$$\log_{-\frac{\pi}{2}}(z) = \ln|z| + i\arg_{-\frac{\pi}{2}}(z) = \ln r + i\theta$$

,

where $z = re^{i\theta}$ and $-\dfrac{\pi}{2} < \theta \le \dfrac{3\pi}{2}$. The path C of integration will consist of the segments $[-R, -r]$ and $[r, R]$ of the x axis together with the upper semicircles $C_r : z = re^{i\theta}$ and $C_R : z = Re^{i\theta}$, for $0 \le \theta \le \pi$, as shown in Figure 8.8.



The contour C for the integrand $f(z) = \dfrac{\log_{-\frac{\pi}{2}}(z)}{z^2 + a^2}$. We chose the branch $\log_{-\frac{\pi}{2}}(z)$ because it is analytic on C and its interior, hence so is the function f(z) which has a simple pole in the upper half-plane at the point $z_1 = a\,i$.

$$\text{Res}[f, z_1] = \text{Res}[f, a\,i]$$

$$= \lim_{z \to a\,i} (z - a\,i)\, f(z)$$

$$= \lim_{z \to a\,i} (z - a\,i) \frac{\log_{-\frac{\pi}{2}}(z)}{(z + a\,i)(z - a\,i)}$$

$$= \lim_{z \to a\,i} \frac{\log_{-\frac{\pi}{2}}(z)}{(z + a\,i)}$$

$$= \frac{\log_{-\frac{\pi}{2}}(a\,i)}{(a\,i + a\,i)} = \frac{\log_{-\frac{\pi}{2}}(a\,i)}{2\,a\,i}$$

$$= \frac{\ln a + i \arg_{-\frac{\pi}{2}}(a\,i)}{2\,a\,i} = \frac{\ln a + i\frac{\pi}{2}}{2\,a\,i}$$

$$= \frac{\ln a}{2\,a\,i} + \frac{\pi}{4\,a}$$

This choice enables us to apply the residue theorem properly, and we get

$$\oint_C f(z)\,dz = 2\pi i \operatorname{Res}[f, z_1] = 2\pi i \operatorname{Res}[f, a i]$$

$$= 2\pi i\left(\frac{\ln a}{2 a i} + \frac{\pi}{4 a}\right)$$

$$= \frac{\pi \ln a}{a} + i\frac{\pi^2}{2 a}$$

Keeping in mind the branch of logarithm that we're using, we then have

$$\oint_C f(z)\,dz = \int_{-R}^{-r} f(x)\,dx + \oint_{-C_r} f(z)\,dz + \int_r^R f(x)\,dx + \oint_{C_R} f(z)\,dz$$

$$= \int_{-R}^{-r} \frac{\ln|x| + i\pi}{x^2 + a^2}\,dx + \oint_{-C_r} f(z)\,dz + \int_r^R \frac{\ln x}{x^2 + a^2}\,dx + \oint_{C_R} f(z)\,dz$$

$$= \frac{\pi \ln a}{a} + i\frac{\pi^2}{2 a}$$

If $R^2 > r^2$, then by the ML inequality

$$\left|\oint_{C_R} f(z)\,dz\right| = \left|\int_0^{\pi} \frac{\ln R + i\theta}{R^2 e^{i2\theta} + a^2}\, i R e^{i\theta}\,d\theta\right|$$

$$\leq \frac{R(\ln R + \pi)\pi}{R^2 - a^2}$$

and L'Hôpital's rule yields $\lim_{R\to\infty} \oint_{C_R} f(z)\,dz = 0$. A similar computation shows that $\lim_{r\to 0} \oint_{C_r} f(z)\,dz = 0$.

We use these results when we take limits Equations to get

$$\text{P.V.}\left(\int_{-\infty}^{0} \frac{\ln|x| + i\pi}{x^2 + a^2}\,dx + \int_{0}^{\infty} \frac{\ln x}{x^2 + a^2}\,dx\right) = \lim_{\substack{R\to\infty \\ r\to 0}}\left(\int_{-R}^{-r} \frac{\ln|x| + i\pi}{x^2 + a^2}\,dx + \int_{r}^{R} \frac{\ln x}{x^2 + a^2}\,dx\right)$$

$$= \oint_C f(z)\,dz$$

$$= \frac{\pi \ln a}{a} + i\,\frac{\pi^2}{2a}$$

Equating the real parts in this equation gives

$$\text{P.V.}\left(\int_{-\infty}^{0} \frac{\ln|x| + i\pi}{x^2 + a^2}\,dx + \int_{0}^{\infty} \frac{\ln x}{x^2 + a^2}\,dx\right) = \frac{\pi \ln a}{a}$$

$$\text{P.V.}\left(\int_{0}^{\infty} \frac{\ln x}{x^2 + a^2}\,dx + \int_{0}^{\infty} \frac{\ln x}{x^2 + a^2}\,dx\right) = \frac{\pi \ln a}{a}$$

$$\text{P.V.}\int_{0}^{\infty} \frac{\ln x}{x^2 + a^2}\,dx = \frac{\pi \ln a}{2a}$$

Assignments:

Evaluation of Real Trigonometric Integrals

$$\int_0^{2\pi} \frac{1}{1+0.5\sin(\theta)} d\theta \quad \int_0^{2\pi} \frac{1}{10-6\cos(\theta)} d\theta$$

$$\int_0^{2\pi} \frac{\cos(\theta)}{3+\sin(\theta)} d\theta \quad \int_0^{2\pi} \frac{1}{1+3\cos^2(\theta)} d\theta$$

$$\int_0^{\pi} \frac{1}{2-\cos(\theta)} d\theta \quad \int_0^{\pi} \frac{1}{1+\sin(\theta)} d\theta$$

$$\int_0^{2\pi} \frac{\sin^2(\theta)}{5+4\cos(\theta)} d\theta \quad \int_0^{2\pi} \frac{\cos^2(\theta)}{3-\sin(\theta)} d\theta$$

Evaluation of Real Improper Integrals

$$\int_{-\infty}^{\infty} \frac{1}{x^2-2x+2} dx, \int_{-\infty}^{\infty} \frac{1}{x^2-6x+25} dx,$$

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^2} dx, \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx,$$

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3} dx, \int_{-\infty}^{\infty} \frac{x}{(x^2+4)^2} dx,$$

Evaluate the Cauchy principal value of the given improper integral.

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)}\,dx, \quad \int_{-\infty}^{\infty} \frac{\cos 2x}{(x^2+1)}\,dx,$$

$$\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+1)}\,dx, \quad \int_{0}^{\infty} \frac{\cos x}{(x^2+4)^2}\,dx,$$

$$\int_{0}^{\infty} \frac{\cos 3x}{(x^2+1)^3}\,dx, \quad \int_{-\infty}^{\infty} \frac{\sin x}{(x^2+4x+5)}\,dx,$$

$$\int_{0}^{\infty} \frac{\cos 2x}{(x^4+1)}\,dx, \qquad \int_{0}^{\infty} \frac{x\sin x}{(x^4+1)}\,dx,$$

RIEVIEW EXECISE 1

Check which statement are true and which are false.

1. $Re(z_1 z_2) = Re(z_1)\, Re(z_2)$

2. $\mathrm{Im}(4 + 7i) = 7i$

3. $|z - 1| = |\bar{z} - 1|$

4. If $\mathrm{Im}(z) > 0$, then $\mathrm{Re}(1/z) > 0$.

5. $i < 10i$

6. If $z \neq 0$, then $Arg(z + \bar{z}) = 0$.

7. $|x + iy| \leq |x| + |y|$

8. $arg(\bar{z}) = arg(1/z)$

9. If $\bar{z} = -z$, then z is pure imaginary.

10. $arg(-2 + 10i) = \pi - tan - 1(5) + 2n\pi$ for n an integer.

11. If z is a root of a polynomial equation $anzn + an - 1zn - 1 + \cdot \cdot \cdot + a1z + a0 = 0$, then $\bar{z}$ is also a root.

12. For any nonzero complex number z, there are an infinite number of values for $\arg(z)$.

13. If $|z - 2| < 2$, then $|\mathrm{Arg}(z)| < \pi/2$.

14. The set S of complex numbers $z = x + iy$ whose real and imaginary parts are related by $y = \sin x$ is a bounded set.

15. The set S of complex numbers z satisfying $|z| < 1$or $|z - 3i| < 1$ is a domain.

16. Consider a set of S of complex numbers. If the set A of all real parts of the numbers in S is bounded and the set B of all imaginary parts of the numbers in S is bounded, then necessarily the set S is bounded.

17. The sector defined by $-\pi/6 < arg(z) \leq \pi/6$ is neither open or nor closed.

18. For $z \neq 0$, there are exactly five values of $z^{\frac{3}{5}} = (z^3)^{\frac{1}{5}}$.

19. A boundary point of a set S is a point in S.

20. The complex plane with the real and imaginary axes deleted has no boundary points.

21. $Im(e^{i\theta}) = sin\,\theta.$

22. The equation $zn = 1$, n a positive integer, will have only real solutions for $n = 1$ and $n = 2$.

In Problems 23–50, try to fill in the blanks without referring back to the text.

23. If $a + ib = \frac{3 - i}{2 + 3i} + \frac{2 - 2i}{1 - 5i}$, then a = --------- and b =----------- .

24. If $z = \frac{4i}{-3 - 4i}$, then $|z|$ =----------- .

25. If $|z|$ = Re(z), then z is ---------.

26. If $z = 3 + 4i$, then $Re(\frac{z}{\bar{z}})$=---------------

27. The principal argument of $z = -1 - i$ is --------------.

28. If z is a point in the second quadrant, then $i\bar{z}$ is in the quadrant.

29. $i^{127} - 5i^9 + 2i - 1$ =-------------- .

30. Of the three points $z1 = 2.5 + 1.9i, z2 = 1.5 - 2.9i$, and $z3 = -2.4 + 2.2i$, is the farthest from the origin.

31. If $3i\bar{z} - 2z = 6$, then z = ---------------.

32. If $2x - 3yi + 9 = -x + 2yi + 5i$, then z = ------------------.

33. If $z = 5 - \sqrt{3} + i$, then Arg(z) = .

34. If $z \neq 0$ is a real number, then $z + z^{-1}$ is real. Other complex numbers $z = x+iy$ for which $z + z^{-1}$ is real are defined by $|z|$ = -------------.

35. The position vector of length 10 passing through (1, −1) is the same as the complex number z =---------------- .

36. The vector $z = (2 + 2i)(\sqrt{3} + i)$ lies in the quadrant.

37. The boundary of the set S of complex numbers z satisfying both Im(y) > 0 and $|z - 3i| > 1$ is .

38. In words, the region in the complex plane for which Re(z) < Im(z) is .

39. The region in the complex plane consisting of the two disks $|z + i| \leq 1$ and $|z - i| \leq 1$ is (connected/not connected).

40. Suppose that $z0$ is not a real number. The circles $|z - z0| = |\overline{Z0} - z0|$ and $|z - \overline{z0}| = |z0 - \overline{z0}|$ intersect on the (real axis/imaginary axis).

41. In complex notation, an equation of the circle with center -1 that passes through $2 - i$ is .

42. A positive integer n for which $(1 + i)^n = 4096$ is n = ------.

43. From $(\cos\theta + i\sin\theta)$^4 $= cos\ 4\theta + i\ sin\ 4\theta$ we get the real trigonometric identities cos 4θ = and sin 4θ = .

RIEVIEW EXERCISE 2

In Problems 1–12, answer true or false. If the statement is false, justify your answer by either explaining why it is false or giving a counterexample; if the statement is true, justify your answer by either proving the statement or citing an appropriate result in this chapter.

1. If a complex function f is differentiable at point z, then f is analytic at z.

2. The function is $f(z) = \frac{y}{x^2+y^2} + \frac{ix}{x^2+y^2}$ differentiable for all $z \neq 0$.

3. The function $f(z) = z^2 + \bar{z}$ is no where analytic.

4. The function $f(z) = \cos y - i \sin y$ is nowhere differentiable.

5. There does not exist an analytic function $f(z) = u(x, y) + iv(x, y)$ for which $u(x, y) = y^3 + 5x$.

6. The function $u(x, y) = e^{4x} \cos 2y$ is the real part ofan analytic function.

7. If $f(z) = e^x \cos y + ie^x \sin y$, then $f(z) = f(\bar{z})$.

8. If u(x, y) and v(x, y) are harmonic functions in a domain D, then the function $f(z) = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} + i(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$ is analytic in D.

9. If g is an entire function, then $f(z) = iz^2 + zg(z)$ is necessarily an entire function.

10. The Cauchy-Riemann equations are necessary conditions for differentiability.

11. The Cauchy-Riemann equations can be satisfied at a point z, but the function $f(z) = u(x, y) + iv(x, y)$ can be non-differentiable at z.

12. If the function $f(z) = u(x, y)+iv(x, y)$ is analytic at a point z, then necessarily the function $g(z) = v(x, y) - iu(x, y)$ is analytic at z.

In Problems 13–22, try to fill in the blanks without referring back to the text.

13. If $f(z) = \frac{1}{z^2} + 5iz - 4$, then $f(z)$ =------------- .

14. The function $f(z) = \frac{1}{z^2} + 5iz - 4$ is not analytic at ------.

15. The function $f(z) = (2 - x)^3 + i(y - 1)^3$ is differentiable at z =-------------- .

16. For $f(z) = 2x^3 + 3iy^2, f(x + ix^2)$= .

17. The function $f(z) = \frac{x-1}{(x-1)^2 + (y-1)^2} - i \frac{y-1}{(x-1)^2 + (y-1)^2}$ is analytic in a domain D not containing the point $z = 1+i$. In D, $f(z)$ =------------ .

18. Find an analytic function $f(z) = loge(x^2 + y^2) + i - - - - - - -$ in a domain D not containing the origin.

19. The function $f(z)$ is analytic in a domain D and $f(z) = c + iv(x, y)$, where c is a real constant. Then f is a------------ in D.

20. $lim z \to 2i \frac{(z^5 - 4iz^4 - 4z^3 + z^2 - 4iz + 4)}{5z^4 - 20iz^3 - 21z^2 - 4iz + 4} = ------$.

21. $u(x, y) = c1$ where $u(x, y) = e^{-x}(x \sin y - y \cos y)$ and $v(x, y) = c2$ where $v(x, y)$ =--------- are orthogonal families.

22. The statement "*There exists a function f that is analytic for* Re(z) ≥ 1 *and is not analytic anywhere else*" is false because-----------------.

Review Exercise 3

In Problems 1–20, answer true or false.If the statement is false, justify your answer by either explaining why it is false or giving a counterexample; if the statement is true, justify your answer by either proving the statement or citing an appropriate result in this chapter.

1. If $|e^z| = 1$, then z is a pure imaginary number.

2. $Re\,(e^z) = cos\,y$.

3. The mapping $w = e^z$ takes vertical lines in the z-plane onto horizontal lines in the w-plane.

4. There are infinitely many solutions z to the equation $e^z = w$.

5. $ln\,i = \frac{1}{2}\pi i$.

6. $Im\,(ln\,z) = arg(z)$.

7. For all nonzero complex z, $e^{Lnz} = z$.

8. If $w1$ and $w2$ are two values of $ln\,z$, then $Re\,(w1) = Re(w2)$.

9. $Ln\frac{1}{z} = -Ln\,z$ for all nonzero z.

10. For all nonzero complex numbers, $Ln\,(z1z2) = Ln\,z1 + Ln\,z2$.

11. Ln z is an entire function.

12. The principal value of $i^i + 1$ is $e^{-\frac{\pi}{2}+i}$.

13. The complex power z^α is always multiple-valued.

14. $\cos z$ is a periodic function with a period of 2π.

15. There are complex z such that $|sin\ z| > 1$.

16. $\tan z$ has singularities at $z = (2n + 1)\,\pi/2$, for $n = 0, \pm 1, \pm 2, \ldots$.

17. $cosh\, z = cos(iz)$.

18. $z = 1/2\ \pi i$ is a zero of $\cosh z$.

19. The function $sin\ \bar{z}$ is nowhere analytic.

20. Every branch of $tan^{-1}\ z$ is entire.

In Problems 21–40, try to fill in the blanks without referring back to the text.

21. The real and imaginary parts of e^z are $u(x, y)$ =----- and $v(x, y)$ =---- .

22. The domain of Ln z is ---- , and its range is ----.

23. $Ln(\sqrt{3} + i) = ------$.

24. The complex exponential function e^z is periodic with a period of ---------.

25. If $e^{iz} = 2$, then $z = -----$.

26. $Ln(e^{1-\pi i}) = -----$.

27. Ln z is discontinuous on -----.

28. The line segment $x = a$, $-\pi < y \leq \pi$, is mapped onto by the mapping $w = e^z$.

29. $ln\ (1 + i) = ----$.

30. If ln z is pure imaginary, then $|z| = -----$.

31. $z1 = 1$ and $z2$ =---- are two real numbers for which the principal value $z^i = 1$.

32. The principal value of i^i is -----.

33. On the domain $|z| > 0, -\pi < arg(z) < \pi$, the derivative of the principal value of z^α is .

34. The complex sine function is defined by sin z = ----.

35. cos $(4i)$ = ----.

36. The semi-infinite vertical strip $-\pi/2 \leq x \leq \pi/2, y \geq 0$, is mapped onto ------ by w = sin z.

37. The real and imaginary parts of $sin\, z$ are ---- and ----, respectively.

38. The complex sine and hyperbolic sine functions are related by the formulas sin(iz) =----- and sinh(iz) =----- .

39. $tanh^{-1} z$ is not defined for z =------ .

40. In order to compute a specific value of $sin^{-1} z$ you need to choose a branch of ----- and a branch of -----.